On the $(2, 1)$-extendability of ternary linear codes

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Overview

The (2, 1)-extendability of \([n, k, d]_3\) codes with \(\gcd(3, d) = 1\) is investigated geometrically for \(3 \leq k \leq 5\).
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1. Geometric approach
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0. Introduction

$C: [n, k, d]_3$ code, $k \geq 3$, $\gcd(d, 3) = 1$.

The diversity $(\Phi_0, \Phi_1)$ of $C$ is given as

$$\Phi_0 = \frac{1}{2} \sum_{3| i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \neq 0, d \text{ (mod 3)}} A_i.$$
$\mathcal{D}_k$ = the set of all possible diversities of $[n, k, d]_3$ codes.

$\mathcal{D}_k^* = \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\}$, where $\theta_j = (3^{j+1} - 1)/2$.

$\mathcal{D}_k^+ = \mathcal{D}_k \setminus \mathcal{D}_k^*$. 
Theorem 1 (Maruta, 2005).

$\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*.$

Problem.

Is $\mathcal{C}$ $(2, 1)$-extendable when $\mathcal{C}$ is not extendable?

We consider this problem for $3 \leq k \leq 5.$

(See Proceedings for $k = 6$)
## Diversities of non-extendable ternary codes

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<thead>
<tr>
<th>$k$</th>
<th>diversity</th>
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<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>(121, 81), (94, 135), (121, 108), (112, 126), (130, 117), (121, 135), (148, 108)</td>
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</table>

How about (2, 1)-extendability of such codes?
1. Geometric approach

$C: [n, k, d]_3$ code, $k \geq 3$

$G = [g_1, \cdots, g_k]^\top$: a generator matrix of $C$

$\Sigma := \text{PG}(k - 1, 3)$: the projective space of dimension $k - 1$ over $\mathbb{F}_3$

For $P = P(p_1, \ldots, p_k) \in \Sigma$ we define the weight of $P$ with respect to $C$, denoted by $w_C(P)$, as

$$w_C(P) = \text{wt}(\sum_{i=1}^{k} p_i g_i).$$
Let

\[ F_d = \{ P \in \Sigma \mid w_C(P) = d \}, \]

\[ F_0 = \{ P \in \Sigma \mid w_C(P) \equiv 0 \pmod{3} \}, \]

\[ F_1 = \{ P \in \Sigma \mid w_C(P) \not\equiv 0, d \pmod{3} \}, \]

\[ F_2 = \{ P \in \Sigma \mid w_C(P) \equiv d \pmod{3} \} \supset F_d, \]

\[ F = F_0 \cup F_1, \quad F_e = F_2 \setminus F_d. \]

Then \((\Phi_0, \Phi_1) = (|F_0|, |F_1|)\).
2. (2,1)-extendability of ternary linear codes

**Lemma 2.** $\mathcal{C}$ is (2,1)-extendable

$\iff \exists (k - 3)$-flat $\subset F \cup F_e$

$\iff \exists (k - 3)$-flat $\subset F = F_0 \cup F_1$.

Every non-extendable $[n,3,d]_3$ code $\mathcal{C}$ with $\gcd(3,d) = 1$ has diversity $(4,3)$.

$\mathcal{C}$ is (2,1)-extendable since $F \neq \emptyset$. 

$\Pi_t$: a $t$-flat in $\Sigma$.

The *diversity of* $\Pi_t$ is $(\varphi_0(t), \varphi_1(t))$

where $\varphi_s(t) = |\Pi_t \cap F_s|$, $s = 0, 1$.

$\Pi_t$ is called a $(\varphi_0(t), \varphi_1(t))_t$ flat.

$(i, j)_1$ flats, $(i, j)_2$ flats, $(i, j)_3$ flats are called $(i, j)$-lines, $(i, j)$-planes, $(i, j)$-solids, respectively.

$c_{i,j}^{(t)}$: the number of $(i, j)_{t-1}$ flats in $\Pi_t$.

The list of $c_{i,j}^{(t)}$'s is called its *spectrum*. 
Table 1.

<table>
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<tr>
<th>$\varphi_0^{(2)}$</th>
<th>$\varphi_1^{(2)}$</th>
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$\exists (i, j)$-line $\Rightarrow \exists (3i + 1, 3j)$-plane.

(4,3) and (4,6) are new.
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(4, 3)-plane

(13, 9)-solid

\[ \delta_1' \]
\[ \delta_2' \]
\[ \delta_3' \]
\[ \delta_1 \]
\[ \delta_2 \]
\[ \delta_3 \]
\[ \delta_4 \]

\[ \delta_5 \]
\[ \delta_6 \]

〇 : a point of \( F_0 \)
□ : a point of \( F_1 \)
● : a point of \( F_2 \)
Theorem 3. Let $C$ be an $[n, k, d]_3$ code with $k = 3$ or 4, $\gcd(3, d) = 1$. Then $C$ is $(2, 1)$-extendable.

Theorem 4. Let $C$ be an $[n, 5, d]_3$ code with diversity $(\Phi_0, \Phi_1) \neq (40, 36)$, $\gcd(3, d) = 1$. Then $C$ is $(2, 1)$-extendable.
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Theorem 5. Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, $\gcd(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable iff either

(a) $\exists l: (1, 0)$-line, $l \subset F_0 \cup F_e$, or

(b) $\exists R_1, R_2, R_3 \in F_e$ s.t. $\langle R_i, R_j \rangle: (0, 2)$-line.
Example. Let $\mathcal{C}$ be a $[15, 5, 8]_3$ code with a generator matrix

$$G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1
\end{bmatrix},$$

whose weight distribution is

$0^{18}6^{40}9^{40}10^{62}11^{20}12^{40}13^{10}14^{10}$ (diversity (40,36)).
(b)

\[ R_1 = (0, 0, 1, 1, 1) \]

\[ R_2 = (1, 2, 1, 2, 2) \]

\[ R_3 = (1, 1, 0, 0, 1) \]

\[ Q_1 \]

\[ Q_2 \]

\[ Q_3 \]

\[ Q_4 \]

\[ Q_5 \]

\[ Q_6 \]

\[ w_C(R_1) = 11, \quad w_C(R_2) = 14, \quad w_C(R_3) = 11. \]

\[ w_C(R_i) \equiv 2 \equiv d = 8 \pmod{3}. \]

\[ w_C(Q_1) = 10, \quad w_C(Q_2) = 10 \cdots \quad w_C(Q_j) \equiv 1 \pmod{3}. \]
Since \( V(x_0 + 2x_1 + 2x_2 + x_3) \cap V(x_0 + x_2 + 2x_4) = \langle R_1, R_2, R_3 \rangle \), by adding the columns \((1, 2, 2, 1, 0)^\top\) and \((1, 0, 1, 0, 2)^\top\) to \(G\), we get a \((2,1)\)-extension of \(C\) whose weight distribution is

\[
0^1 9^{38} 10^{56} 11^{46} 12^{34} 13^{30} 14^{26} 15^8 16^4.
\]

**Remark.**

We have no example of an \([n, 5, d]_3\) code with diversity \((40,36)\), \(\text{gcd}(3,d) = 1\), which is not \((2,1)\)-extendable.
Theorem 6. Let $C$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, $\gcd(3, d) = 1$. Then $C$ is $(2, 1)$-extendable if $A_d \leq 50$.

To prove this, we need:

(1) In $(40, 36)_4$ flat, any $(10, 15)$-solid has the unique focal point from $F_1$ and any $(16, 12)$-solid has the unique focal point from $F_2$.

(2) Every 20-cap in $PG(4, 3)$ is either a $\Gamma$-cap or $\Delta$-cap.
3. Focal points and focal hyperplanes

For \( i = 1, 2 \), a point \( P \in F_i \) is called a focal point of a hyperplane \( H \) (or \( P \) is focal to \( H \)) if the following three conditions hold:

(a) \( \langle P, Q \rangle \) is a \( (0, 2) \)-line for \( Q \in F_i \cap H \),
(b) \( \langle P, Q \rangle \) is a \( (2, 1) \)-line for \( Q \in F_{3-i} \cap H \),
(c) \( \langle P, Q \rangle \) is a \( (1, 6 - 3i) \)-line for \( Q \in F_0 \cap H \).
Such a hyperplane $H$ is called a *focal hyperplane* of $P$ (or $H$ is *focal to* $P$).

**Theorem 7**
Let $\Pi_t$ be a $t$-flat with new diversity. Then, for $i = 1, 2$, any point of $\Pi_t \cap F_i$ has a focal $(a, b)$-hyperplane in $\Pi_t$ for some new $(a, b)$.

**Note.**
Let $(2,1)$ and $(0,2)$ be new in the types of lines for convenience.
(a) \( \langle P, Q \rangle \) is a (0, 2)-line for \( Q \in F_2 \cap l \)
(b) \( \langle P, Q \rangle \) is a (2, 1)-line for \( Q \in F_1 \cap l \)
(c) \( \langle P, Q \rangle \) is a (1, 0)-line for \( Q \in F_0 \cap l \)
(a) $\langle P, Q \rangle$ is a $(0, 2)$-line for $Q \in F_2 \cap l$

(b) $\langle P, Q \rangle$ is a $(2, 1)$-line for $Q \in F_1 \cap l$

(c) $\langle P, Q \rangle$ is a $(1, 0)$-line for $Q \in F_0 \cap l$
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(c) $\langle P, Q \rangle$ is a $(1, 0)$-line for $Q \in F_0 \cap l$
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(c) \( \langle P, Q \rangle \) is a \((1, 0)\)-line for \(Q \in F_0 \cap l\)
Lemma 8. Let $\delta$ be a $(4, 3)$-plane. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(0, 2)$-line and a focal $(2, 1)$-line, respectively, and vice versa.
Lemma 9. Let $\delta$ be a $(4,6)$-plane. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(2,1)$-line and a focal $(0,2)$-line, respectively, and vice versa.
Lemma 10. Let $\delta$ be a $(16,12)$-solid. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(4,3)$-plane and a focal $(4,6)$-plane, respectively, and vice versa.
\( (16, 12) \)-solid

\( (4, 3) \)-plane

\( (4, 6) \)-plane
Lemma 11. Let $\Pi$ be a $(40, 36)_4$ flat. Then, every point of $\Pi \cap F_1$ and of $\Pi \cap F_2$ has a focal $(10, 15)$-solid and a focal $(16, 12)$-solid, respectively, and vice versa.

Assume $\mathcal{C}$ is not $(2,1)$-extendable. Then no three points of $F_e$ are collinear by Theorem 5.

$F_e$ forms a cap and we have $|F_e| \leq 20$ since the largest size of a cap in $PG(4,3)$ is 20.

Every 20-cap in $PG(4,3)$ is either a $\Gamma$-cap or a $\Delta$-cap (Pellegrino, 1974).
\[ \Gamma\text{-}cap = \bigcup_{i=1}^{10} \{ \text{two of } \Box \in VP_i \}. \]
$\triangle-cap = \{\bullet\} \cup \{\bigcirc\} \cup \{\star\} \cup \{\bigstar\} \cup \{\bigtriangleup\} \cup \{\blacklozenge\}$
We only consider the case when $F_e$ is a 20-cap of type $\Gamma$. (See Proceedings for the case when $F_e$ is a 20-cap of type $\Delta$.)
Case 1: $V \in F_0$

Actually, there are exactly six $(1,0)$-lines through $V$ in the $(40,36)$ flat $\Sigma$, a contradiction.
Case 1: $V \in F_0 \Rightarrow VP_i = (1, 0)$-line

Actually, there are exactly six $(1, 0)$-lines through $V$ in the $(40, 36)_4$ flat $\Sigma$, a contradiction.
Case 2: $V \in F_1$
Case 2: $V \in F_1 \Rightarrow VP_i = (0, 2)$-line
$H'$: the focal $(10,15)$-solid of $V$

$E'$: the projection of $E$ from $V$ onto $H'$ if $H \neq H'$. 
$V$ is point of $F_1 \Rightarrow VP_i = (0,2)$-line

In $H'$, at most $c_{1,6}^{(3)} + c_{4,6}^{(3)} = 25$ planes meet $E'$ in four points, a contradiction.
Case 3: \( V \in F_2 \Rightarrow VP_i = (1, 0)\)-line
Thank you for your attention!