On the $(2,1)$-extendability of ternary linear codes

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Overview

♣ The $(2,1)$-extendability of $[n,k,d]_3$ codes with $\gcd(3,d)=1$ is investigated geometrically for $3 \leq k \leq 5$.

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0. Introduction

$\mathcal{C}$: $[n,k,d]_3$ code, $k \geq 3$, $\gcd(d,3)=1$.

The diversity (Φ_0, Φ_1) of $\mathcal{C}$ is given as

$$\Phi_0 = \frac{1}{2} \sum_{3|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv 0, d \pmod 3} A_i.$$

$\mathcal{D}_k$ = the set of all possible diversities of $[n, k, d]_3$ codes.

$$\mathcal{D}_k^* = \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\},$$

where $\theta_j = (3^{j+1} - 1)/2$.

$\mathcal{D}_k^+ = \mathcal{D}_k \setminus \mathcal{D}_k^*$.

Theorem 1 (Maruta, 2005).
$\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*$.

Probrem.
Is $\mathcal{C}$ $(2, 1)$-extendable when $\mathcal{C}$ is not extendable?

We consider this problem for $3 \leq k \leq 5$.
(See Proceedings for $k = 6$)

Diversities of non-extendable ternary codes

k	$diversity$
3	$(4, 3)$
4	$(13, 9), (10, 15), (16, 12)$
5	$(40, 27), (31, 45), (40, 36), (40, 45), (49, 36)$
6	$(121, 81), (94, 135), (121, 108), (112, 126),$ $(130, 117), (121, 135), (148, 108)$

How about $(2, 1)$-extendability of such codes?

1. Geometric approach

$\mathcal{C}$: $[n,k,d]_3$ code, $k \geq 3$

$G = [g_1, \cdots, g_k]^{\mathsf{T}}$: a generator matrix of $\mathcal{C}$

$\Sigma :=$PG$(k-1,3)$: the projective space of dimension $k-1$ over $\mathbb{F}_3$

For $P = \mathbf{P}(p_1, \ldots, p_k) \in \Sigma$ we define the weight of P with respect to $\mathcal{C}$, denoted by $w_{\mathcal{C}}(P)$, as

$$w_{\mathcal{C}}(P) = wt(\sum_{i=1}^{k} p_i g_i).$$

Let

$$\begin{aligned}
F_d &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) = d\}, \\
F_0 &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) \equiv 0 \pmod 3\}, \\
F_1 &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) \not\equiv 0, d \pmod 3\}, \\
F_2 &= \{P \in \Sigma \mid w_{\mathcal{C}}(P) \equiv d \pmod 3\} \ (\supset F_d), \\
F &= F_0 \cup F_1, \ \ F_e = F_2 \setminus F_d.
\end{aligned}$$

Then $(\Phi_0, \Phi_1) = (|F_0|, |F_1|)$.

2. (2,1)-extendability of ternary linear codes

Lemma 2. $\mathcal{C}$ is $(2,1)$-extendable
$\Leftrightarrow \exists(k-3)$-flat $\subset F \cup F_e$
$\Leftarrow \exists(k-3)$-flat $\subset F = F_0 \cup F_1$.

Every non-extendable $[n,3,d]_3$ code $\mathcal{C}$ with $\gcd(3,d)=1$ has diversity $(4,3)$.
$\mathcal{C}$ is $(2,1)$-extendable since $F \neq \emptyset$.

Π_t : a t-flat in Σ.

The *diversity of* Π_t is $(\varphi_0{}^{(t)}, \varphi_1{}^{(t)})$

where $\varphi_s{}^{(t)} = |\Pi_t \cap F_s|$, $s = 0, 1$.

Π_t is called a $(\varphi_0{}^{(t)}, \varphi_1{}^{(t)})_t$ flat.

$(i,j)_1$ flats, $(i,j)_2$ flats, $(i,j)_3$ flats are called (i,j)-lines, (i,j)-planes, (i,j)-solids,respectively.

$c_{i,j}^{(t)}$: the number of $(i,j)_{t-1}$ flats in Π_t.

The list of $c_{i,j}^{(t)}$'s is called its *spectrum*.

Table 1.

$\varphi_0^{(2)}$	$\varphi_1^{(2)}$	$c_{1,0}^{(2)}$	$c_{0,2}^{(2)}$	$c_{2,1}^{(2)}$	$c_{1,3}^{(2)}$	$c_{4,0}^{(2)}$
4	0	12	0	0	0	1
1	6	2	9	0	2	0
4	3	4	3	6	0	0
4	6	0	3	6	4	0
7	3	1	0	9	1	2
4	9	0	0	0	12	1
13	0	0	0	0	0	13

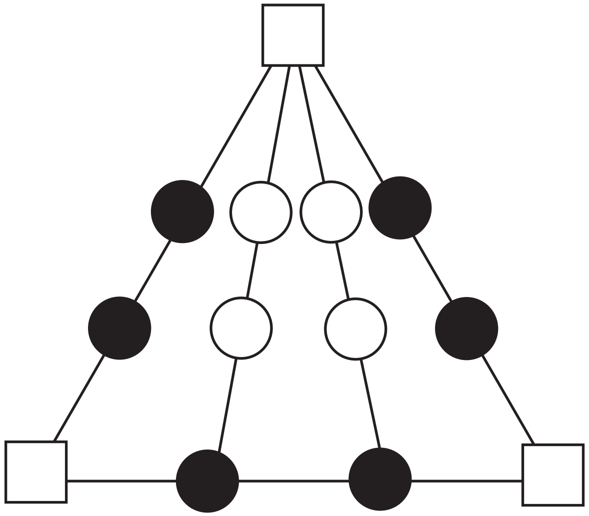
$\exists(i,j)$-line $\Rightarrow \exists(3i+1,3j)$-plane.

$(4,3)$ and $(4,6)$ are new.

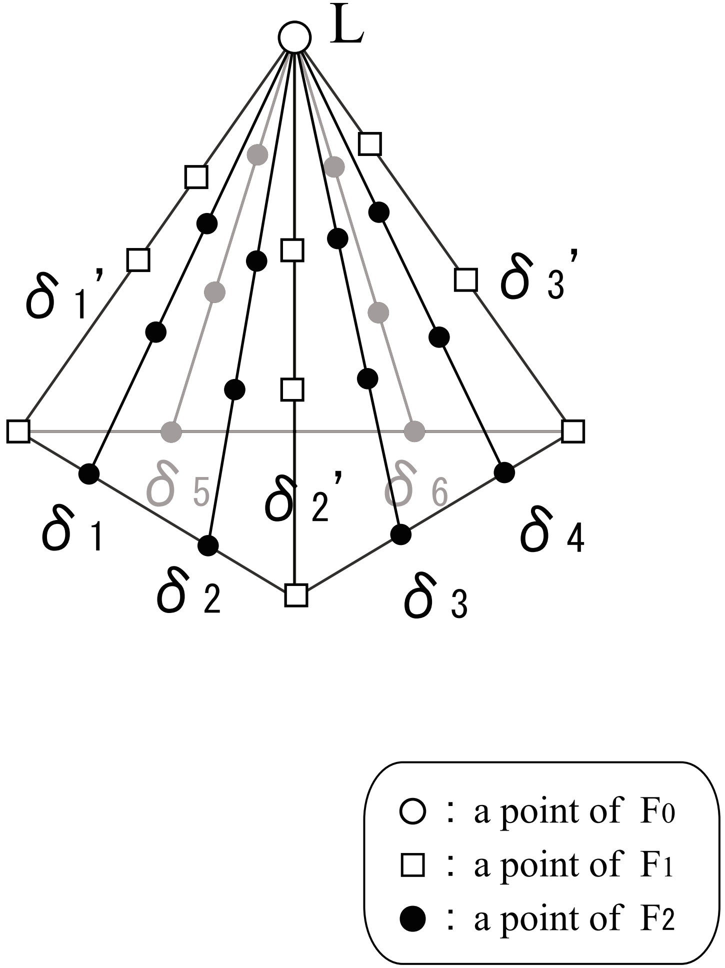
Table 2.

${\varphi_0}^{(3)}$	${\varphi_1}^{(3)}$	$c^{(3)}_{4,0}$	$c^{(3)}_{1,6}$	$c^{(3)}_{4,3}$	$c^{(3)}_{4,6}$	$c^{(3)}_{7,3}$	$c^{(3)}_{4,9}$	$c^{(3)}_{13,0}$
13	0	39	0	0	0	0	0	1
4	18	2	36	0	0	0	2	0
13	9	4	3	27	0	6	0	0
10	15	0	10	15	15	0	0	0
16	12	0	0	12	12	16	0	0
13	18	0	3	0	27	6	4	0
22	9	1	0	0	0	36	1	2
13	27	0	0	0	0	0	39	1
40	0	0	0	0	0	0	0	40

(4, 3)-plane



(13, 9)-solid



Theorem 3. Let $\mathcal{C}$ be an $[n, k, d]_3$ code with $k = 3$ or 4, $\gcd(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable.

Theorem 4. Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(\Phi_0, \Phi_1) \neq (40, 36)$, $\gcd(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable.

Table 3.

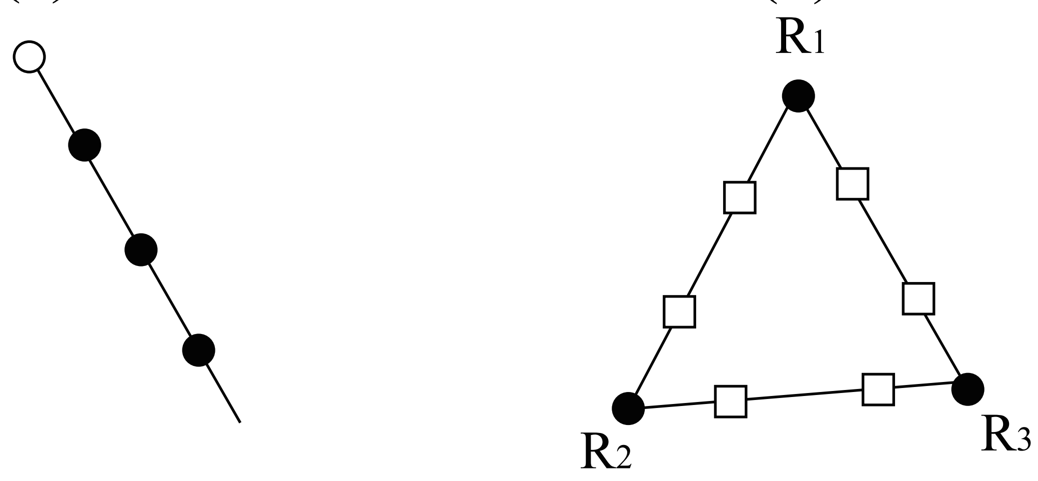
${\varphi_0}^{(4)}$	${\varphi_1}^{(4)}$	$c^{(4)}_{13,0}$	$c^{(4)}_{4,18}$	$c^{(4)}_{13,9}$	$c^{(4)}_{10,15}$	$c^{(4)}_{16,12}$	$c^{(4)}_{13,18}$	$c^{(4)}_{22,9}$	$c^{(4)}_{13,27}$	$c^{(4)}_{40,0}$
40	0	120	0	0	0	0	0	0	0	1
13	54	2	117	0	0	0	0	0	2	0
40	27	4	3	108	0	0	0	6	0	0
31	45	0	10	15	81	0	15	0	0	0
40	36	0	0	40	36	45	0	0	0	0
40	45	0	0	0	36	45	40	0	0	0
49	36	0	0	12	0	81	12	16	0	0
40	54	0	3	0	0	0	108	6	4	0
67	27	1	0	0	0	0	0	117	1	2
40	81	0	0	0	0	0	0	0	120	1
121	0	0	0	0	0	0	0	0	0	121

Theorem 5. Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, $\gcd(3, d) = 1$.
Then $\mathcal{C}$ is $(2, 1)$-extendable iff either

(a) $\exists l$:$(1, 0)$-line, $l \subset F_0 \cup F_e$, or

(b) $\exists R_1, R_2, R_3 \in F_e$ s.t. $\langle R_i, R_j \rangle$:$(0, 2)$-line.

(a) (b)

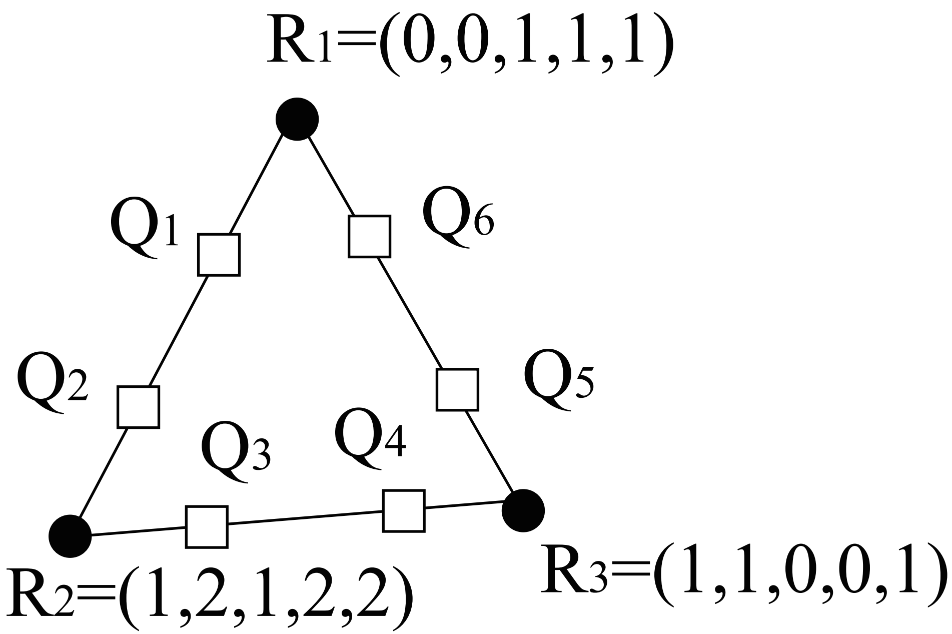


Example. Let $\mathcal{C}$ be a $[15, 5, 8]_3$ code with a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \end{bmatrix},$$

whose weight distribution is
$0^1 8^{60} 9^{40} 10^{62} 11^{20} 12^{40} 13^{10} 14^{10}$ (diversity (40,36)).

(b)



$w_{\mathcal{C}}(R_1) = 11$, $w_{\mathcal{C}}(R_2) = 14$, $w_{\mathcal{C}}(R_3) = 11$.

$w_{\mathcal{C}}(R_i) \equiv 2 \equiv d = 8 \pmod 3$.

$w_{\mathcal{C}}(Q_1) = 10$, $w_{\mathcal{C}}(Q_2) = 10 \cdots$. $w_{\mathcal{C}}(Q_j) \equiv 1 \pmod 3$.

Since $V(x_0+2x_1+2x_2+x_3) \cap V(x_0+x_2+2x_4) = \langle R_1, R_2, R_3 \rangle$, by adding the columns $(1,2,2,1,0)^{\mathsf{T}}$ and $(1,0,1,0,2)^{\mathsf{T}}$ to G, we get a (2,1)-extension of $\mathcal{C}$ whose weight distribution is

$$0^1 9^{38} 10^{56} 11^{46} 12^{34} 13^{30} 14^{26} 15^8 16^4.$$

Remark.

We have no example of an $[n,5,d]_3$ code with diversity (40,36), $\gcd(3,d)=1$, which is not (2,1)-extendable.

Theorem 6. Let $\mathcal{C}$ be an $[n, 5, d]_3$ code with diversity $(40, 36)$, gcd$(3, d) = 1$. Then $\mathcal{C}$ is $(2, 1)$-extendable if $A_d \le 50$.

To prove this, we need:

(1) In $(40, 36)_4$ flat, any $(10, 15)$-solid has the unique focal point from F_1 and any $(16, 12)$-solid has the unique focal point from F_2.

(2) Every 20-cap in $PG(4, 3)$ is either a Γ-*cap* or Δ-*cap*.

3. Focal points and focal hyperplanes

For $i = 1, 2$, a point $P \in F_i$ is called a *focal point* of a hyperplane H (or P is *focal to* H) if the following three conditions hold:

(a) $\langle P, Q \rangle$ is a $(0, 2)$-line for $Q \in F_i \cap H$,

(b) $\langle P, Q \rangle$ is a $(2, 1)$-line for $Q \in F_{3-i} \cap H$,

(c) $\langle P, Q \rangle$ is a $(1, 6 - 3i)$-line for $Q \in F_0 \cap H$.

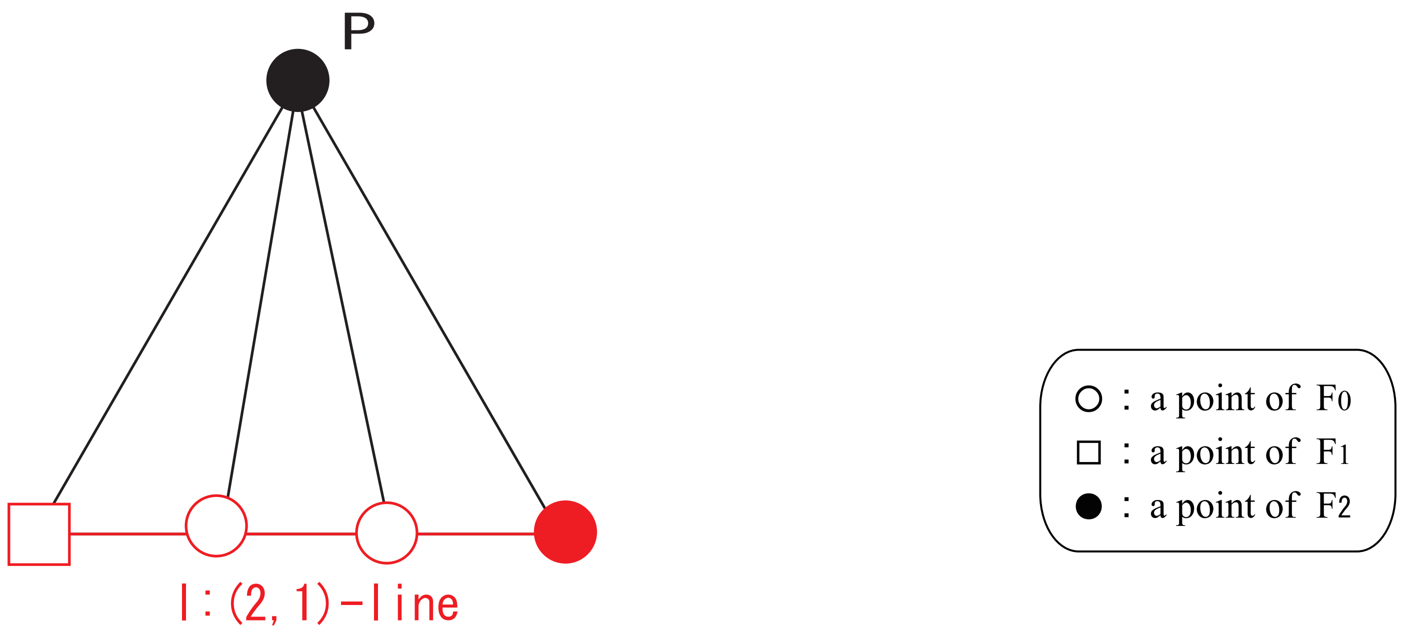
Such a hyperplane H is called a *focal hyperplane* of P (or H is *focal to* P).

Theorem 7

Let Π_t be a t-flat with new diversity. Then, for $i = 1, 2$, any point of $\Pi_t \cap F_i$ has a focal (a, b)-hyperplane in Π_t for some new (a, b).

Note.

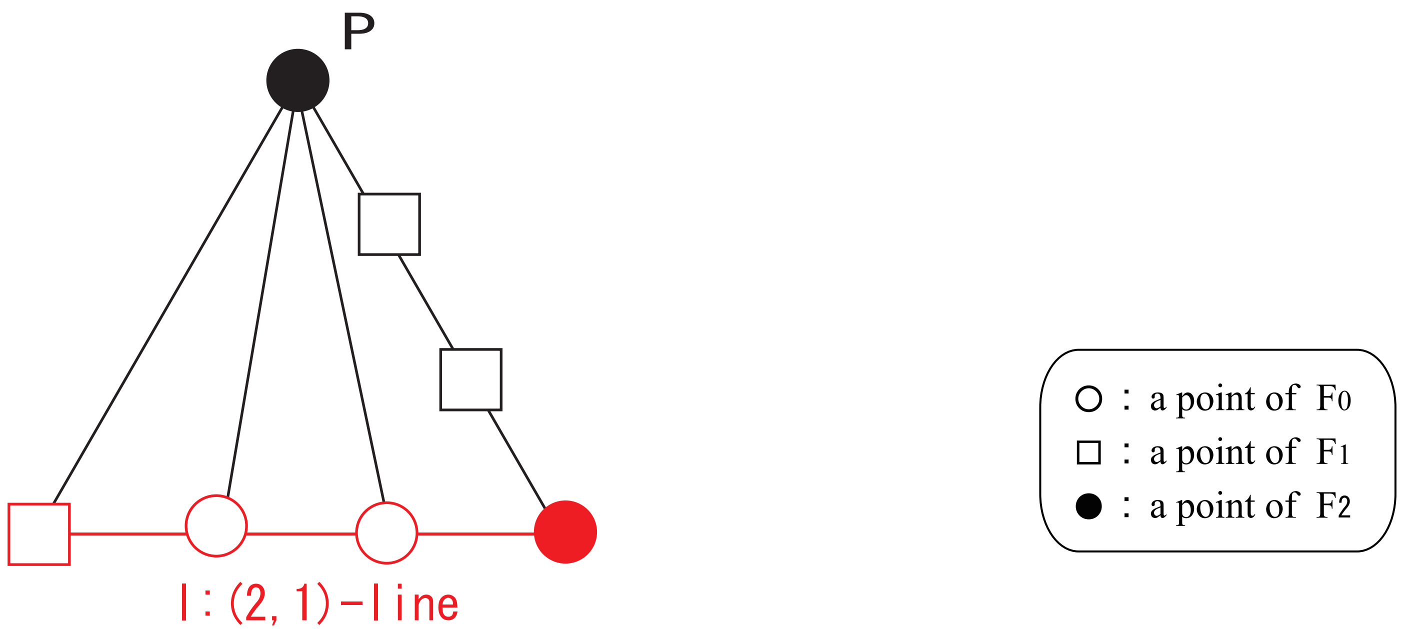
Let (2,1) and (0,2) be new in the types of lines for convenience.



(a) $\langle P, Q\rangle$ is a $(0,2)$-line for $Q \in F_2 \cap l$

(b) $\langle P, Q\rangle$ is a $(2,1)$-line for $Q \in F_1 \cap l$

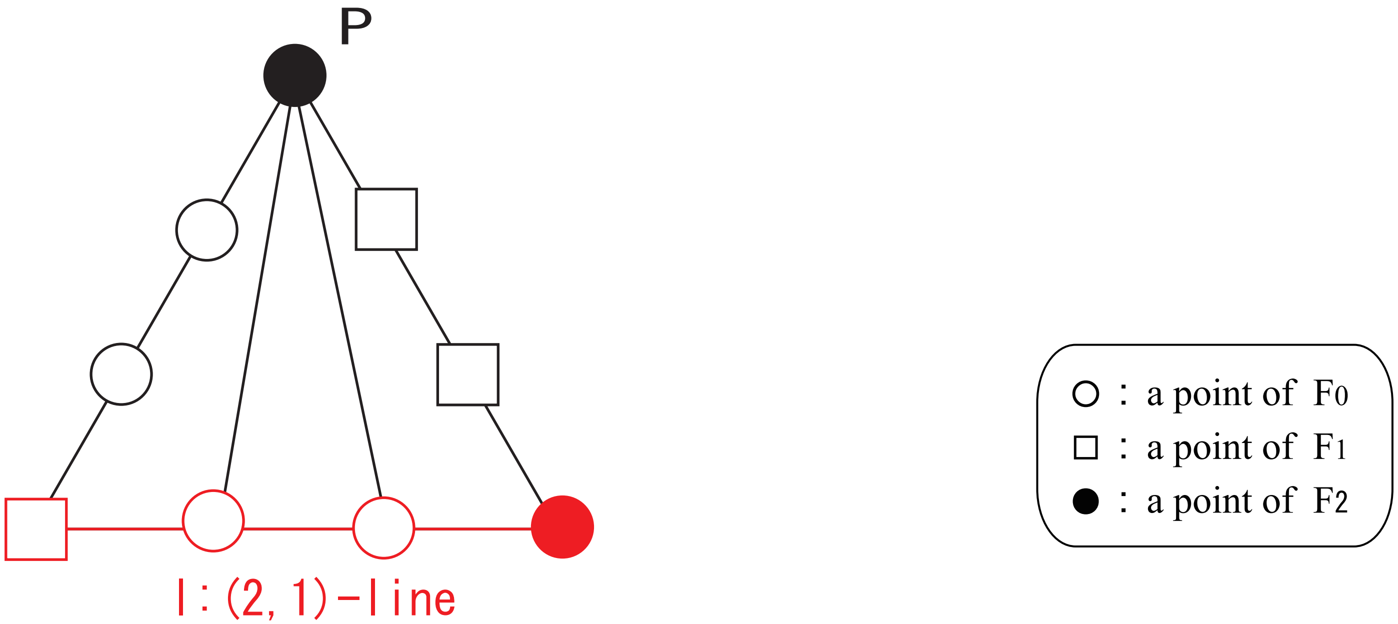
(c) $\langle P, Q\rangle$ is a $(1,0)$-line for $Q \in F_0 \cap l$



(a) $\langle P, Q \rangle$ is a $(0, 2)$-line for $Q \in F_2 \cap l$

(b) $\langle P, Q \rangle$ is a $(2, 1)$-line for $Q \in F_1 \cap l$

(c) $\langle P, Q \rangle$ is a $(1, 0)$-line for $Q \in F_0 \cap l$



(a) $\langle P, Q \rangle$ is a $(0, 2)$-line for $Q \in F_2 \cap l$

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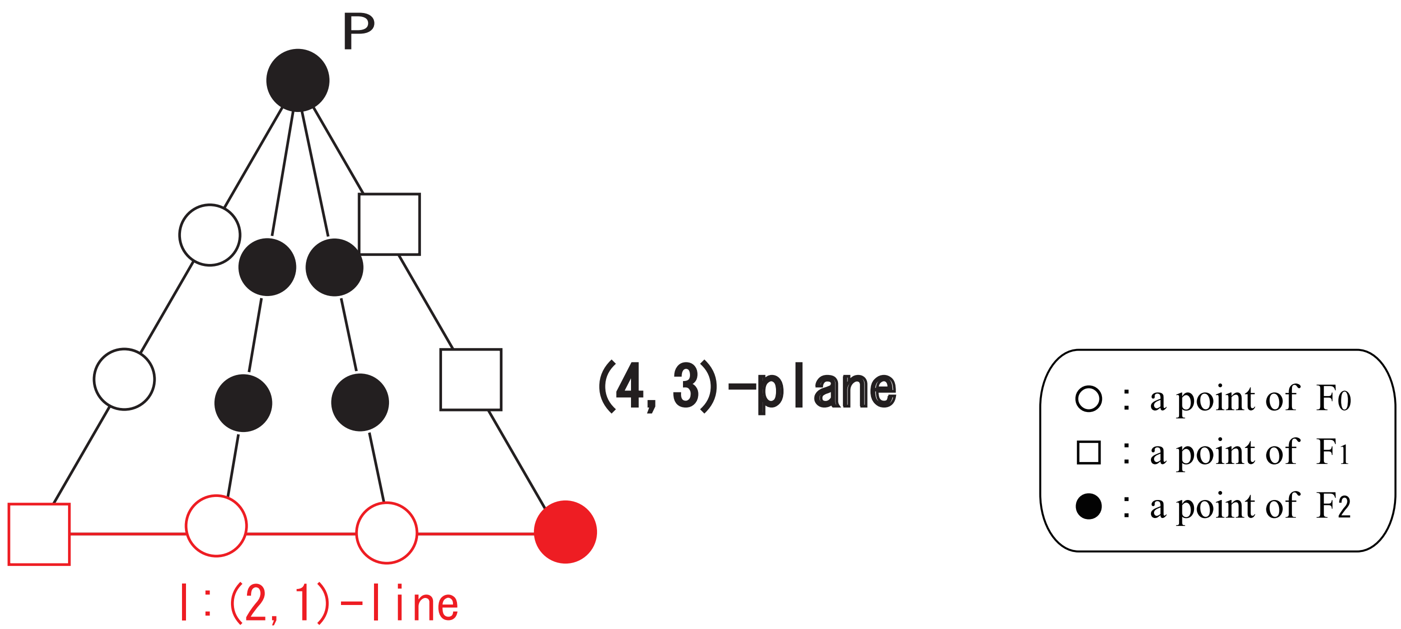
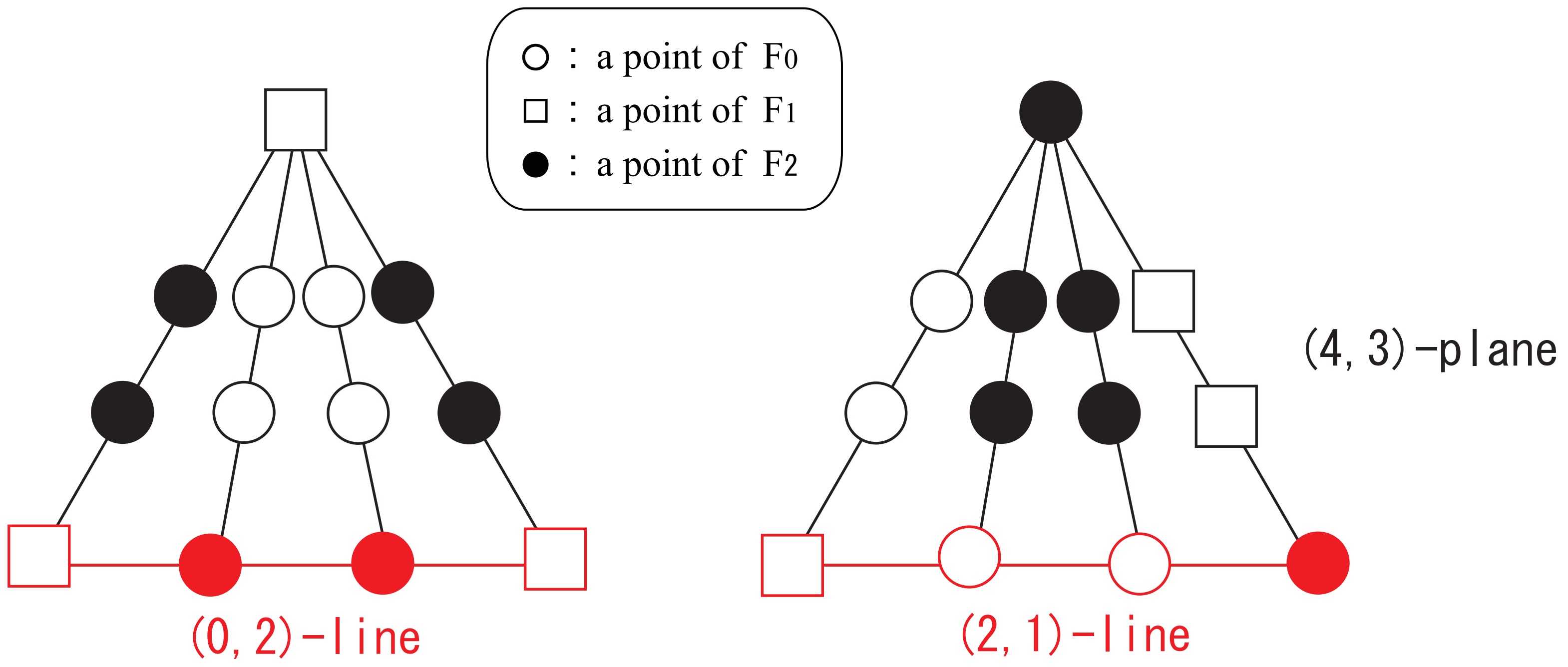


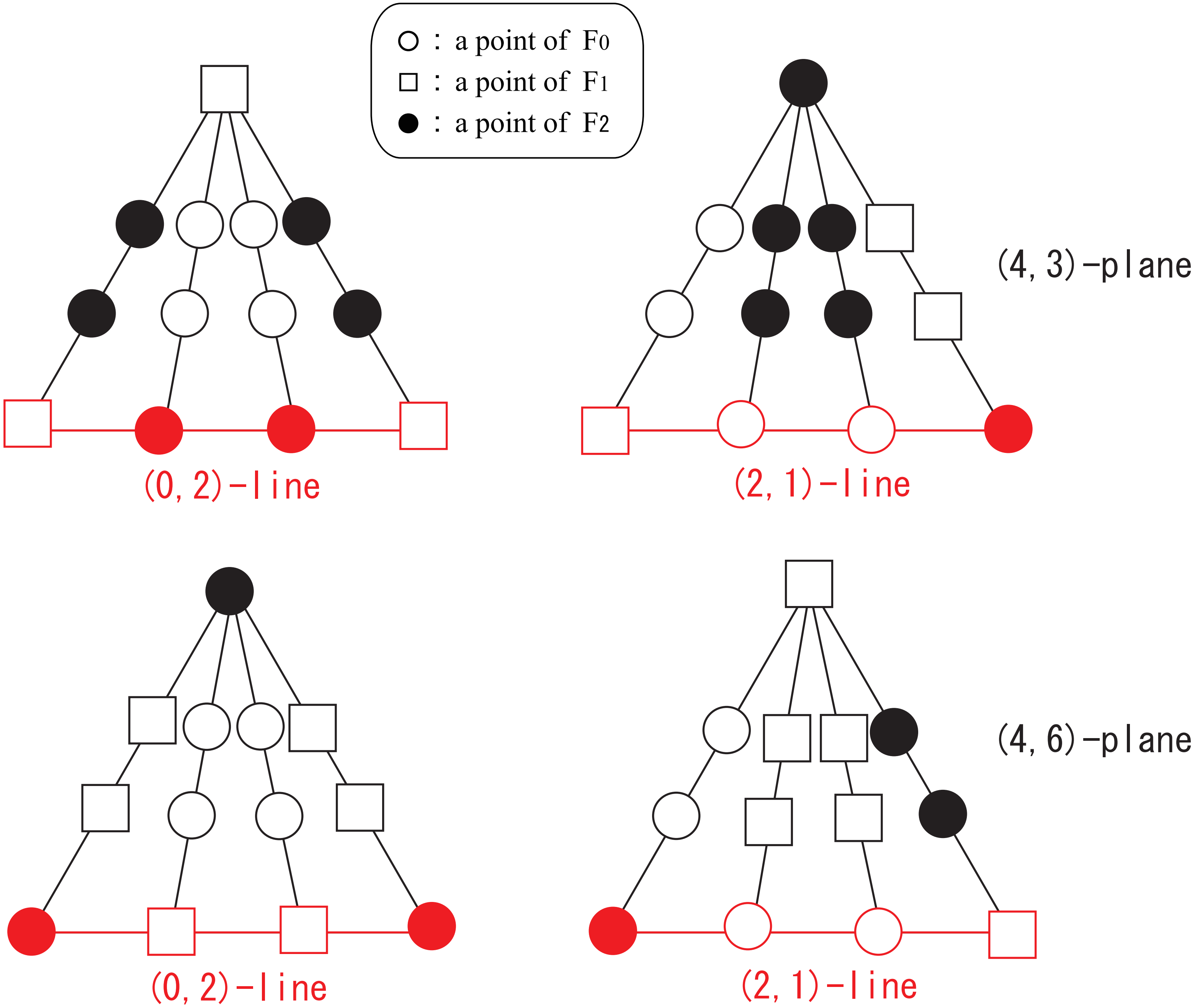
(a) $\langle P, Q\rangle$ is a $(0,2)$-line for $Q \in F_2 \cap l$

(b) $\langle P, Q\rangle$ is a $(2,1)$-line for $Q \in F_1 \cap l$

(c) $\langle P, Q\rangle$ is a $(1,0)$-line for $Q \in F_0 \cap l$

Lemma 8. Let δ be a $(4, 3)$-plane. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(0, 2)$-line and a focal $(2, 1)$-line, respectively, and vice versa.

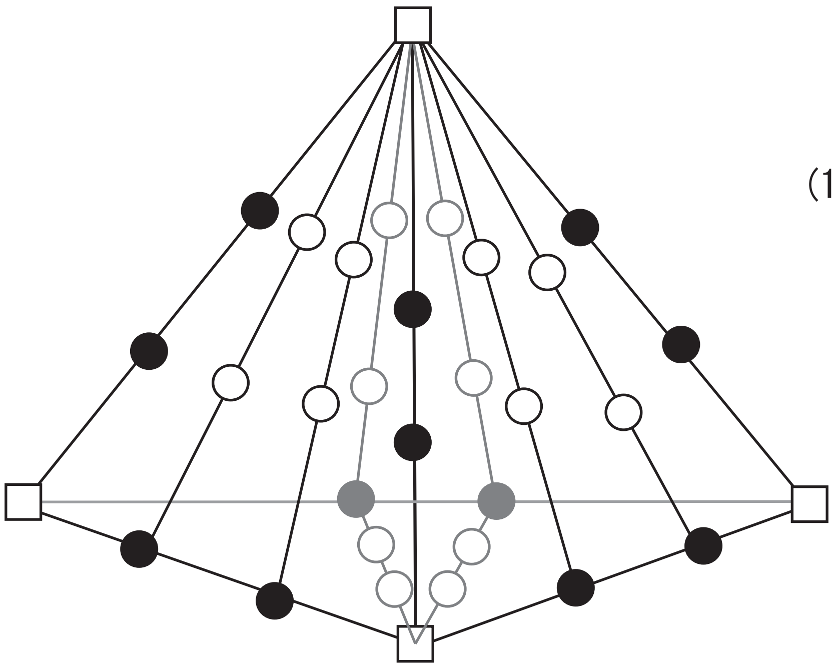

○ : a point of F0
□ : a point of F1
● : a point of F2
(4, 3)-plane
(0, 2)-line
(2, 1)-line


○ : a point of F0
□ : a point of F1
● : a point of F2
(0, 2)-line
(2, 1)-line
(4, 3)-plane
(0, 2)-line
(2, 1)-line
(4, 6)-plane

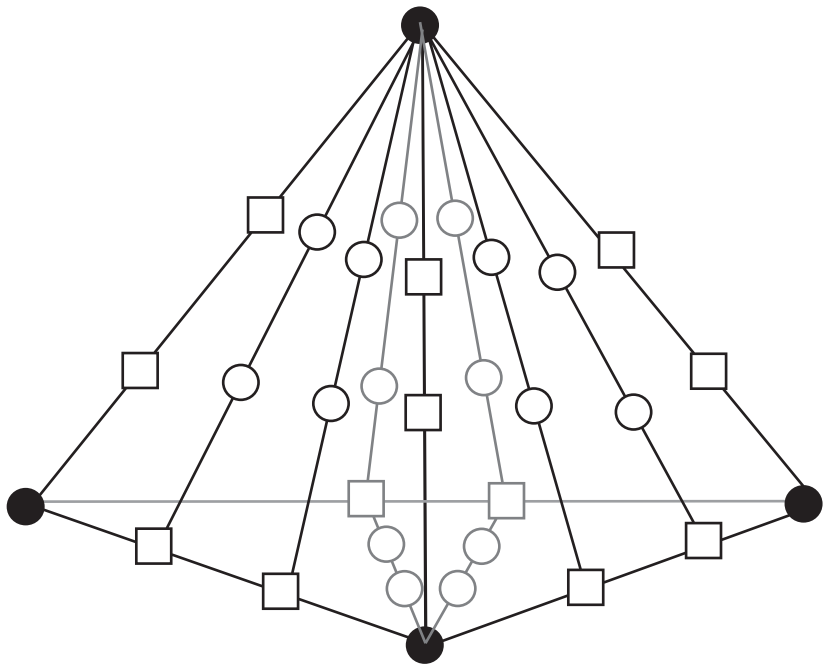
Lemma 9. Let δ be a $(4, 6)$-plane. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(2, 1)$-line and a focal $(0, 2)$-line, respectively, and vice versa.

Lemma 10. Let δ be a $(16, 12)$-solid. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(4, 3)$-plane and a focal $(4, 6)$-plane, respectively, and vice versa.

(16,12)-solid



(4,3)-plane



(4,6)-plane

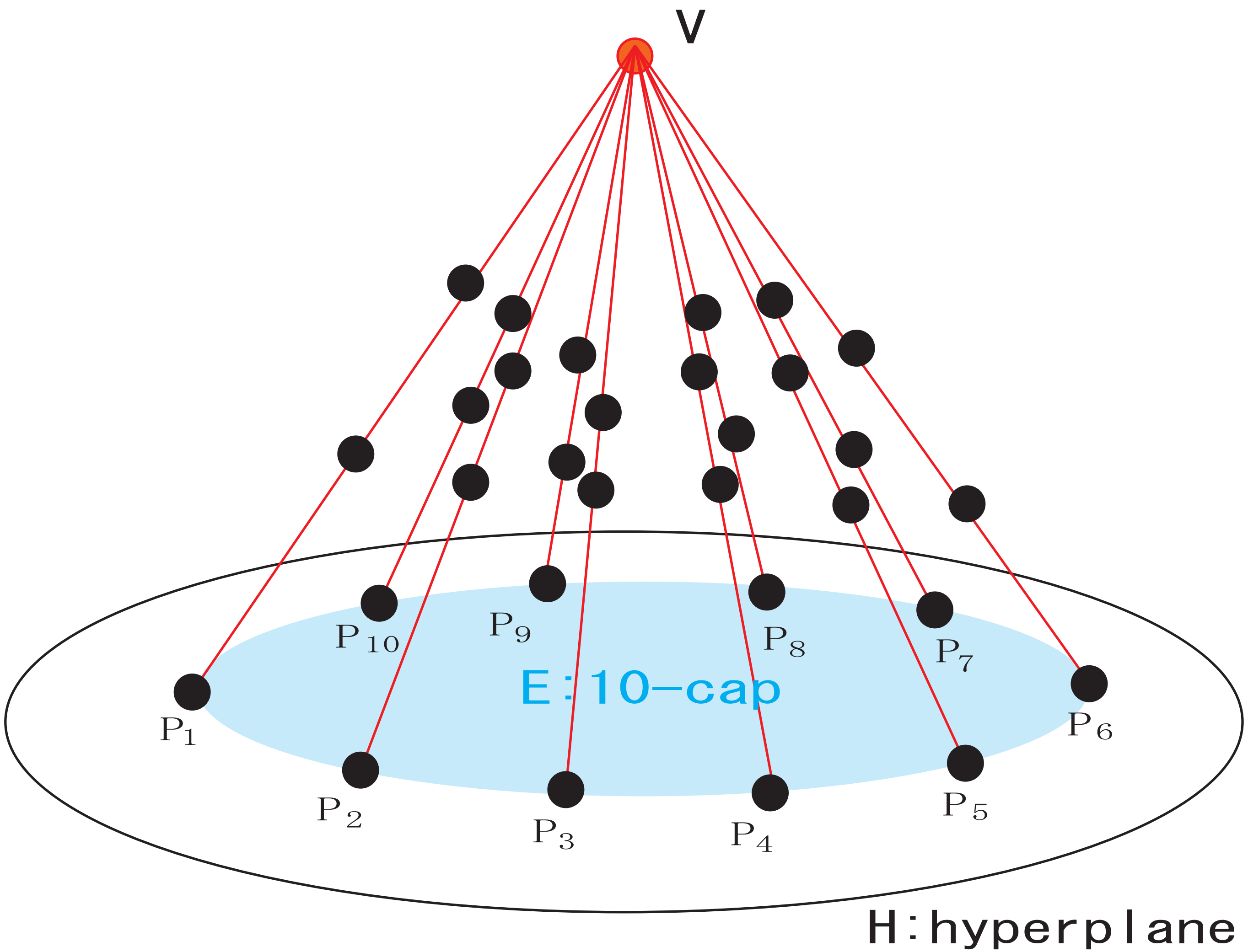
Lemma 11. Let Π be a $(40, 36)_4$ flat. Then, every point of $\Pi \cap F_1$ and of $\Pi \cap F_2$ has a focal $(10, 15)$-solid and a focal $(16, 12)$-solid, respectively, and vice versa.

4. Proof of Theorem 6.

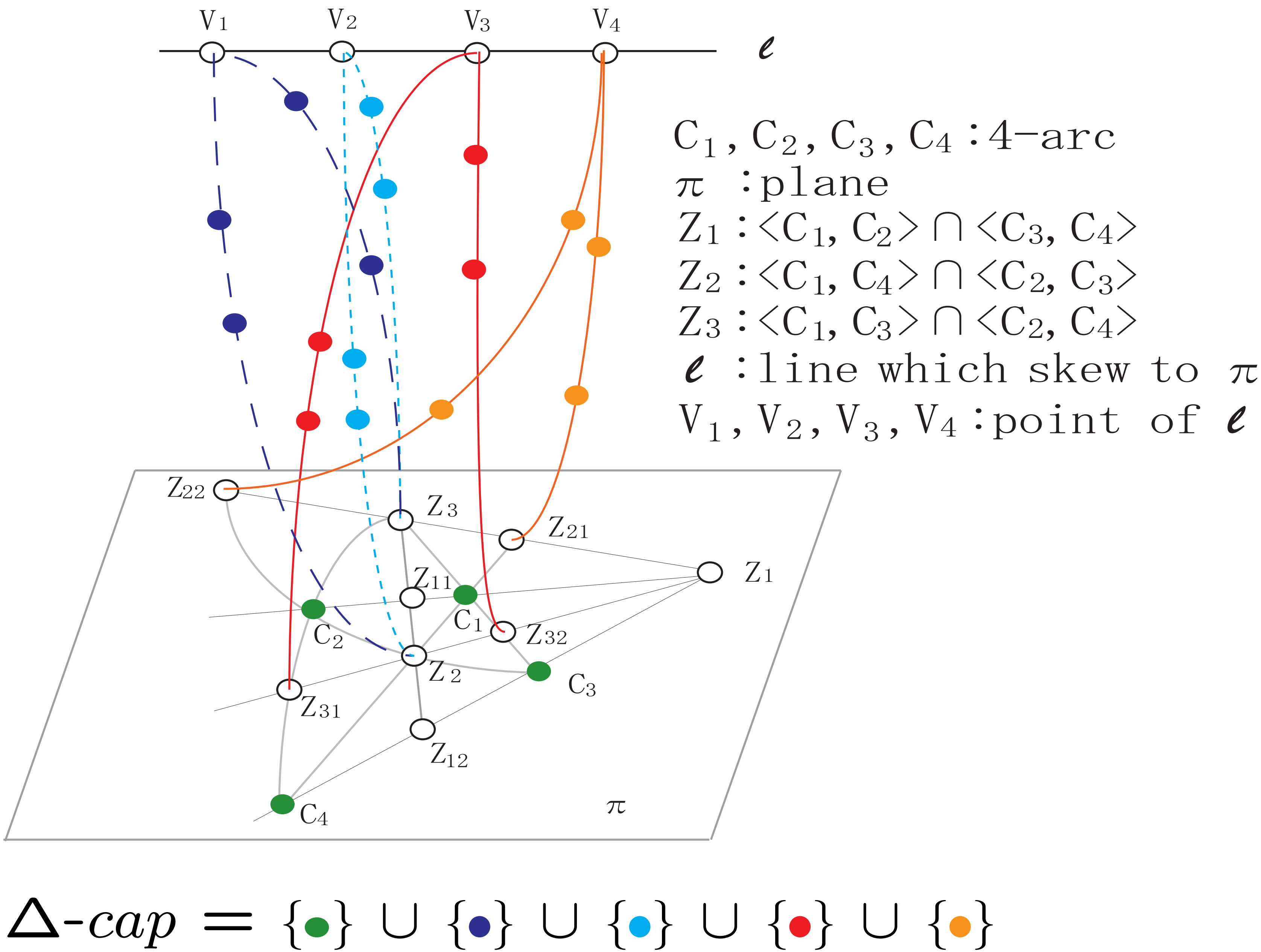
Assume $\mathcal{C}$ is not $(2,1)$-extendable. Then no three points of F_e are collinear by Theorem 5.

F_e forms a cap and we have $|F_e| \leq 20$ since the largest size of a cap in $PG(4,3)$ is 20.

Every 20-cap in $PG(4,3)$ is either a Γ-cap or a Δ-cap (Pellegrino, 1974).



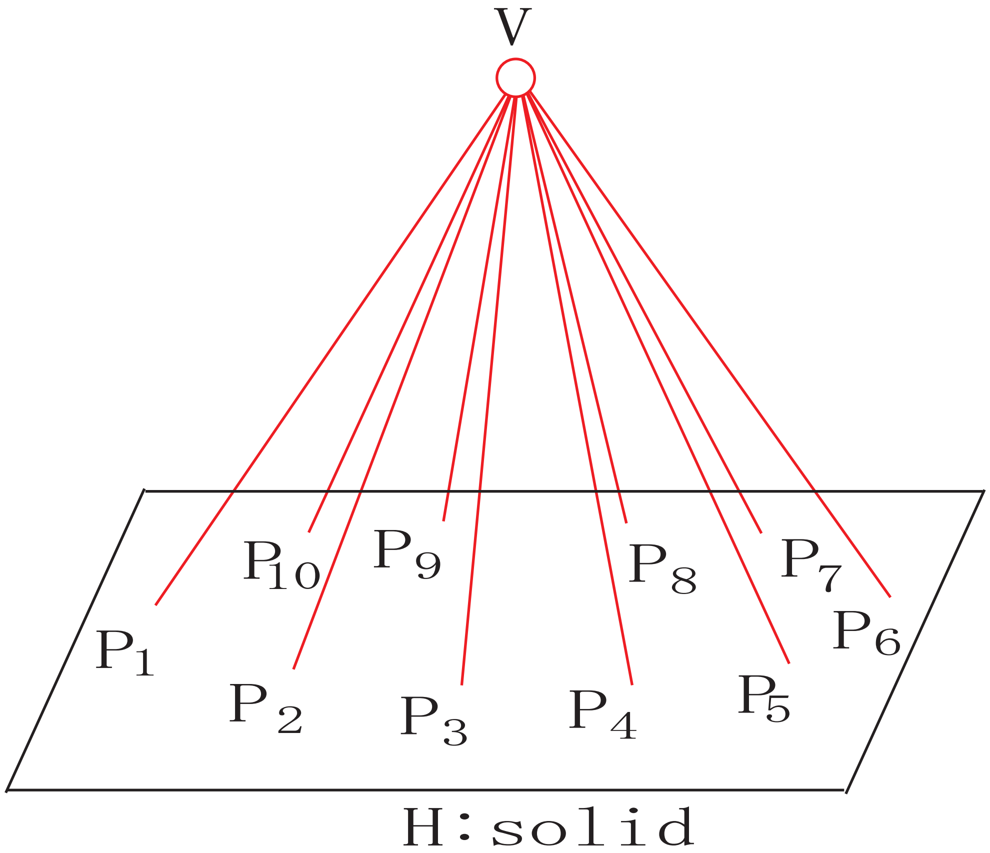
Γ-$cap = \cup_{i=1}^{10}$ { two of ● $\in VP_i$ }.



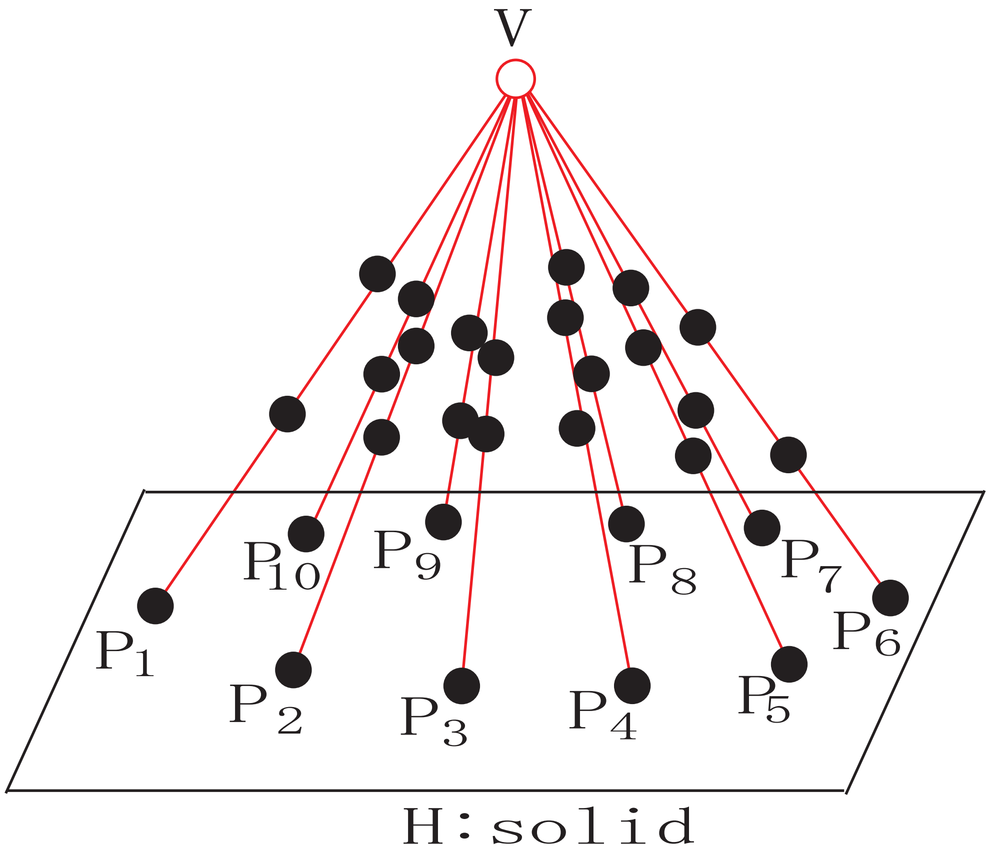
V1
V2
V3
V4
ℓ
C1 , C2 , C3 , C4 :4-arc
π :plane
Z1 :<C1, C2> ∩ <C3, C4>
Z2 :<C1, C4> ∩ <C2, C3>
Z3 :<C1, C3> ∩ <C2, C4>
ℓ :line which skew to π
V1 , V2 , V3 , V4 :point of ℓ
Z22
Z3
Z21
Z1
Z11
C1
C2
Z32
Z2
C3
Z31
Z12
C4
π
Δ-cap = {•} ∪ {•} ∪ {•} ∪ {•} ∪ {•}

We only consider the case when F_e is a 20-cap of type Γ. (See Proceedings for the case when F_e is a 20-cap of type Δ.)

Case 1: $V \in F_0$

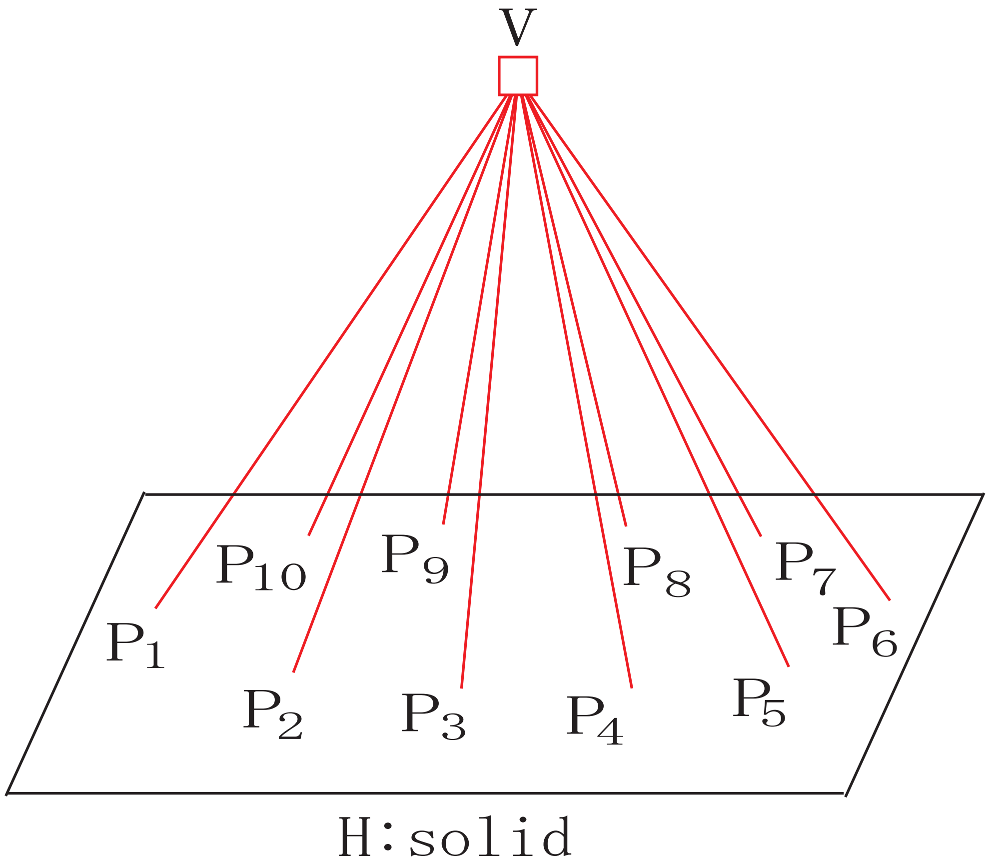


Case 1: $V \in F_0 \Rightarrow VP_i = (1,0)$-line

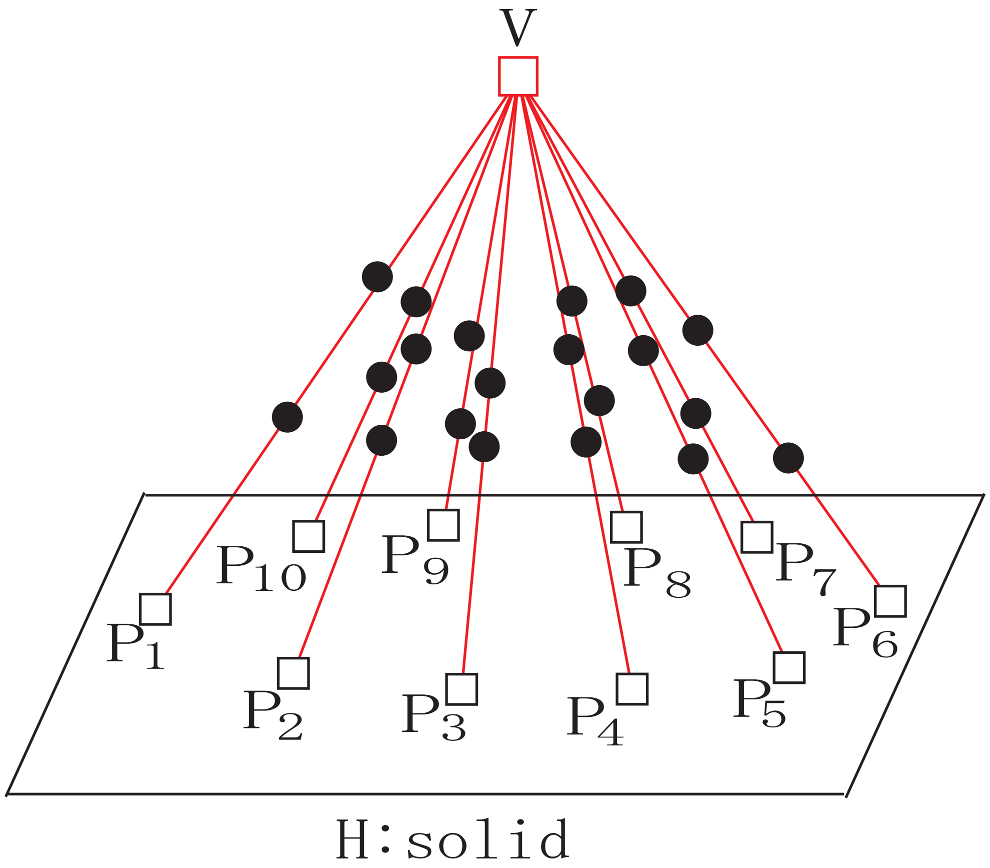


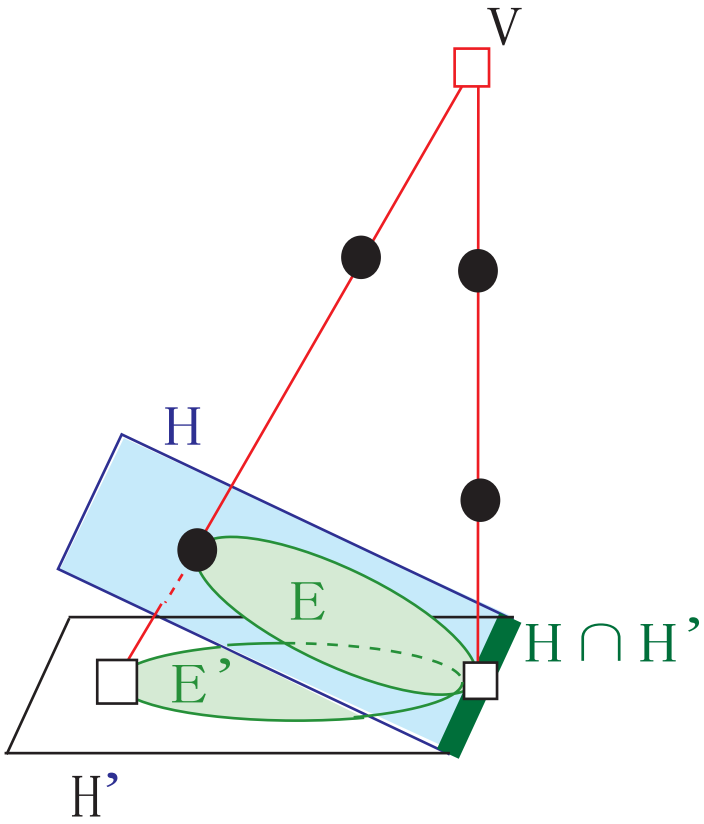
Actually, there are exactly six $(1,0)$-lines through V in the $(40,36)_4$ flat Σ, **a contradiction**.

Case 2: $V \in F_1$



Case 2: $V \in F_1 \Rightarrow VP_i = (0,2)$-line

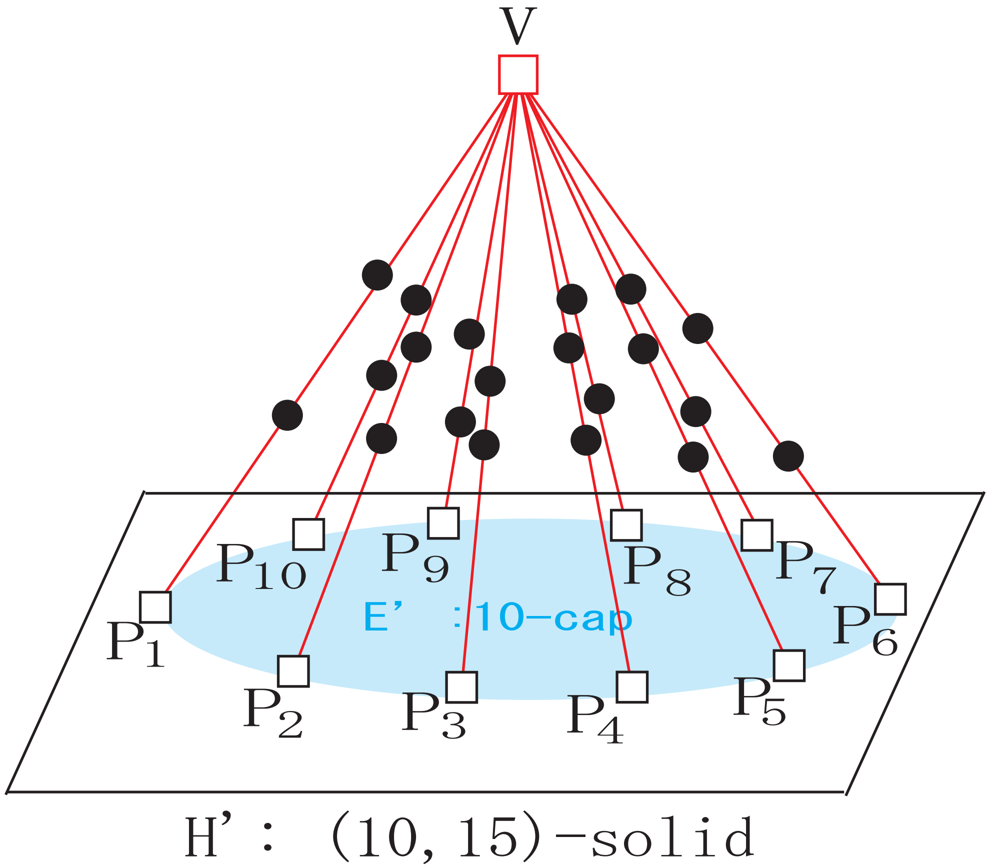




H': the focal $(10, 15)$-solid of V

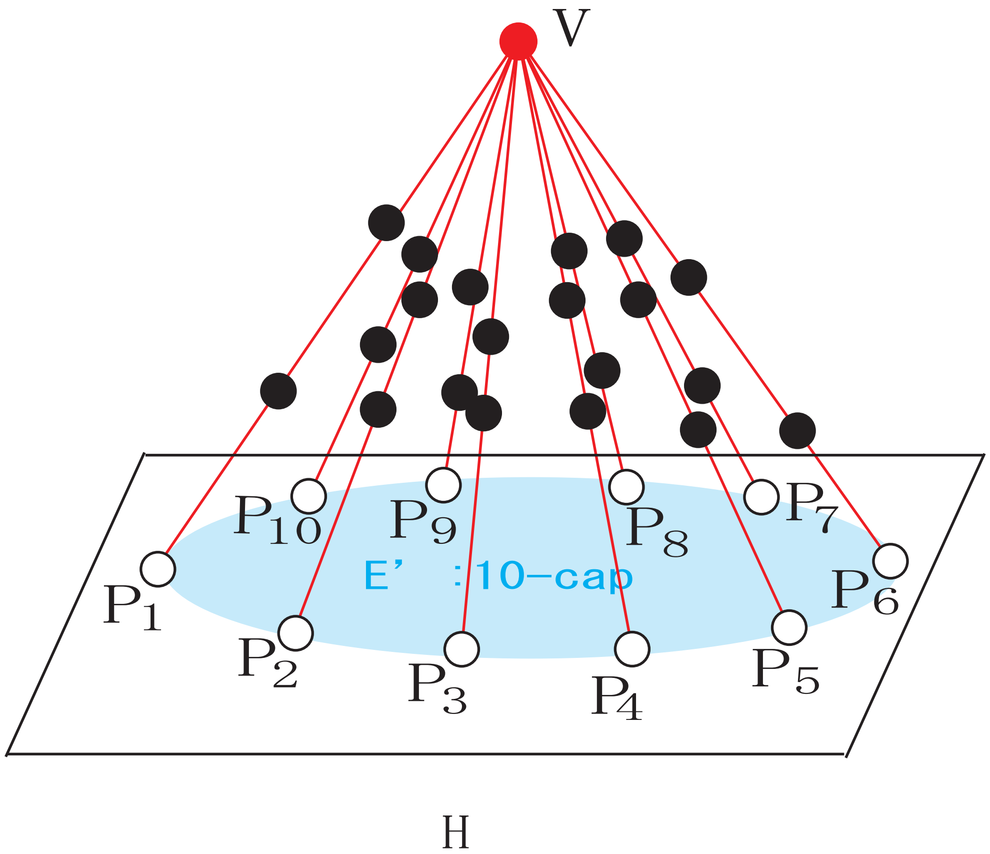
E': the projection of E from V onto H' if $H \neq H'$.

V is point of $F_1 \Rightarrow VP_i = (0,2)$-line



In H', at most $c^{(3)}_{1,6} + c^{(3)}_{4,6} = 25$ planes meet E' in four points, **a contradiction**.

Case 3: $V \in F_2 \Rightarrow VP_i = (1,0)$-line



Thank you for your attention!