Burst Error Correcting Codes and Lattice Paths

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Sum Arrays

$$
\begin{array}{r|rrrrrr}
-2 & -13 & -45 & -105 & -165 & -99 & 429 \\
-2 & -11 & -32 & -60 & -60 & 66 & 528 \\
-2 & -9 & -21 & -28 & 0 & 126 & 462 \\
-2 & -7 & -12 & -7 & 28 & 126 & 336 \\
-2 & -5 & -5 & 5 & 35 & 98 & 210 \\
-2 & -3 & 0 & 10 & 30 & 63 & 112 \\
-2 & -1 & 3 & 10 & 20 & 33 & 49 \\
-2 & 1 & 4 & 7 & 10 & 13 & 16 \\
 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}
$$

$$\beta(n,k) = 3 \cdot \binom{n+k}{k} - 2 \cdot \binom{n+k}{k-1}$$

numbers around the 0s have closed formula

$$\frac{1}{5n+1}\binom{5n+1}{2n} : 2, 5, 60, \ldots; \qquad \frac{1}{5n+2}\binom{5n+2}{2n} : 1, 5, 66, \ldots$$

$$\frac{1}{5n+3}\binom{5n+3}{2n+1} : 1, 7, 99, \ldots; \qquad \frac{3}{5n+1}\binom{5n+1}{2n+1} : 3, 10, 126, \ldots$$

$$\frac{2}{5n-1}\binom{5n-1}{2n} : 3, 28, 429$$

Berlekamp's Problem

Berlekamp (1968, 2000)

$$
\begin{array}{rrrrrrrrr}
 & & & & & & & 1 & 1 \\
 & & & & & & 1 & 1 & 2 \\
 & & & & 1 & 1 & 1 & 2 & 5 \\
 & & & 1 & 1 & 2 & 3 & 7 & 19 \\
 & 1 & 1 & 1 & 2 & 5 & 9 & 23 & 66 \\
1 & 1 & 2 & 3 & 7 & 19 & 37 & 99 & 293
\end{array}
$$

Berlekamp defines an array to be unitary if any square submatrix whose upper left corner falls on the boundary of the array has a determinant equal to 1. For instance, in the array above

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & 9 \end{pmatrix} = 1$$

Problem: Find exact formulas

History

Berlekamp 1963: burst-error correcting convolutional codes, numbers reduced modulo 2 give a generating sequence for such codes

Berlekamp 1968: presents above combinatorial problem at Waterloo conference

Berlekamp 2000: studies above sum array:" The patterns are clear, but I know no explanation. Why does the formula apply to an individual entry, then to sums of pairs of entries, and then to the negative of an entry? I have not found any simple formulas for the individual entries within each pair which is summed together in the above formulas."

Carlitz, Rosselle, Scoville 1971: lattice path approach

T. 2003, 2008: generating function

Recursion via Lattice Paths

					293
				66	293
				66	227
			19	66	161
		5	19	47	95
		5	14	28	48
	2	5	9	14	20
1	2	3	4	5	6
1	1	1	1	1	1

					136
					136
				37	136
			9	37	99
			9	28	62
		3	9	19	34
	1	3	6	10	15
	1	2	3	4	5
1	1	1	1	1	1

					99
				23	99
				23	76
			7	23	53
		2	7	16	30
		2	5	9	14
	1	2	3	4	5
1	1	1	1	1	1

enumerate lattice paths below a line of slope $\frac{3}{2}$ (or $\frac{2}{3}$)

Generating Function, T. 2002, 2008

$$a(x) = g^{(1,3,1)}(x) = \sum_{n=0}^{\infty} \frac{1}{5n+1}\binom{5n+1}{2n}x^n - \frac{x}{2}\cdot[h^{(1,3,1)}(x)]^2$$

$$= 1 + 2x + 23x^2 + 377x^3 + \ldots$$

$$b(x) = g^{(1,3,2)}(x) = \sum_{n=0}^{\infty} \frac{1}{5n+1}\binom{5n+1}{2n}x^n + \frac{x}{2}\cdot[h^{(1,3,1)}(x)]^2$$

$$= 1 + 3x + 37x^2 + 624x^3 + \ldots$$

$$g^{(2,3,1)}(x) = \sum_{n=0}^{\infty} \frac{1}{5n+2}\binom{5n+2}{2n+1}x^n$$

$$= 1 + 5x + 66x^2 + 1156x^3 + \ldots = a(x)\cdot b(x)$$

$$h^{(1,3,1)}(x) = \sum_{n=0}^{\infty} \frac{1}{5n+3}\binom{5n+3}{2n+1}x^n$$

$$= 1 + 7x + 99x^2 + 1768x^3 + \ldots = a(x)^2\cdot b(x)$$

$$h^{(1,3,2)}(x) = \sum_{n=1}^{\infty} \frac{1}{5n-1}\binom{5n-1}{2n}x^{n-1} - \frac{1}{2}[g^{(2,3,1)}(x)]^2$$

$$= 1 + 9x + 136x^2 + \ldots = a(x)^3\cdot b(x)$$

$$h^{(2,3,1)}(x) = \sum_{n=1}^{\infty} \frac{1}{5n-1}\binom{5n-1}{2n}x^{n-1} + \frac{1}{2}\cdot[g^{(2,3,1)}(x)]^2$$

$$= 2 + 19x + 293x^2 + 5332x^3 + \ldots$$

$$= h^{(1,3,2)}(x) + [g^{(2,3,1)}(x)]^2 = a(x)^3\cdot b(x) + a(x)^2\cdot b(x)^2$$

$$= (a(x)+b(x))\cdot a(x)^2\cdot b(x) = (g^{(1,3,1)}(x)+g^{(1,3,2)}(x))\cdot h^{(1,3,1)}(x)$$

Perodic Staircase Boundaries

Chapman, Chow, Khetan, Moulton, Waters; JCT A, to appear:

					126	659
					126	533
					126	407
			10	45	126	281
			10	35	81	155
			10	25	46	74
1	3	6	10	15	21	28
1	2	3	4	5	6	7
1	1	1	1	1	1	1

closed formula for paths starting in (0,0) or in (1,0)

Proof via Reflection Principle

new proof via generating function

Gessel's Probabilistic Approach

We shall consider paths in an integer lattice from the origin $(0,0)$ to the point (n, u_n), which never touch any of the points (m, u_m), $m = 0, 1, \ldots, n-1$.
u_n periodic, for instance, $u_0 = 2$, $u_1 = 3$, $u_2 = 5$, $u_3 = 6$, $u_4 = 8$, $u_5 = 9$, etc.

For period length 2 the elements of the sequence $(u_m)_{m=0,1,2,\ldots}$ are on the 2 lines (for $i = 0, 1, 2, \ldots$)

$$u_{2i} = s + ci \text{ and } u_{2i+1} = s + \mu + ci,$$

Gessel's probabilistic method is as follows. A particle starts at the origin $(0,0)$ and successively moves with probability p one unit to the right and with probability $q = 1 - p$ one unit up. The particle stops if it touches one of the points (i, u_i).
The probability that the particle stops at (n, u_n) is $p^n q^{u_n} \cdot f_n$.
Setting

$$f(t) = \sum_{n=0}^{\infty} f_n t^n = \sum_{n=0}^{\infty} f_{2n} t^{2n} + \sum_{n=0}^{\infty} f_{2n+1} t^{2n+1} = g(t^2) + t \cdot h(t^2)$$

the probability that the particle eventually stops is

$$q^{u_0} g(p^2 q^c) + p q^{u_1} h(p^2 q^c)$$

If p is sufficiently small, the particle will touch the boundary $(m, u_m)_{m=0,1,\ldots}$ with probability 1. So for small p and with $t = pq^{c/d}$ we have

$$q(t)^{u_0} g(t^2) + p(t) q(t)^{u_1} h(t^2) = 1$$

For p sufficiently small one may invert $t = p(1-p)^{c/d}$ to express p as a power series in t, namely $p = p(t)$. Then changing t to $-t$ and denoting $p(-t)$ by $\overline{p}(t)$ and similarly $q(-t)$ by $\overline{q}(t)$ yields the system of equations

$$q^s \cdot g(t^2) + p \cdot q^{s+\mu} \cdot h(t^2) = 1,$$

$$\overline{q}^s \cdot g(t^2) + \overline{p} \cdot \overline{q}^{s+\mu} \cdot h(t^2) = 1$$

which for $g(t^2)$ and $h(t^2)$ yield the solutions

$$g(t^2) = \frac{p^{-1} q^{-s-\mu} - \overline{p}^{-1} \overline{q}^{-s-\mu}}{p^{-1} q^{-\mu} - \overline{p}^{-1} \overline{q}^{-\mu}} = \frac{q^{c/2-\mu-s} + \overline{q}^{c/2-\mu-s}}{q^{c/2-\mu} + \overline{q}^{c/2-\mu}}$$

and

$$h(t^2) = \frac{q^{-s} - \overline{q}^{-s}}{t \cdot (q^{\mu-c/2} + \overline{q}^{\mu-c/2})}$$

By Lagrange inversion for any α we have

$$q^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha}{(c/2+1)n + \alpha} \binom{(c/2+1)n + \alpha}{n} \cdot t^n$$

Open Problems

1) formulas for further entries?

2) period length ≥ 3?

period length 1: Catalan numbers and generalizations

period length 2: our generating functions

period length ≥ 3: solve larger system of equations

$$\begin{array}{ccccccc} q(t)^{u_0} f^{(0)}(t^d) & + & p(t)q(t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(t)^{d-1} q(t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1 \\ q(\omega t)^{u_0} f^{(0)}(t^d) & + & p(\omega t)q(\omega t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(\omega t)^{d-1} q(\omega t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1 \\ \vdots & + & \vdots & + \quad + & \vdots & = \vdots \\ q(\omega^{d-1} t)^{u_0} f^{(0)}(t^d) & + & p(\omega^{d-1} t)q(\omega^{d-1} t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(\omega^{d-1} t)^{d-1} q(\omega^{d-1} t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1 \end{array} .$$

Written in matrix form where $A = (p(\omega^i t)^j q(\omega^i t)^{u_j})_{i,j=0,\ldots,d-1}$ this is

$$A \cdot \begin{pmatrix} f^{(0)}(t^d) \\ f^{(1)}(t^d) \\ \vdots \\ f^{(d-1)}(t^d) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} .$$

1. I still like this problem.

I think relations to coding theory are explained in my 1963 Inf & Control paper. I was a grad student then & was discouraged from pursuing this topic by one of my advisors (Prof. J. Wozencraft), but I've always doubted his rationale.

My goal then was to find structured ways of designing generators for Convolutional Codes. The erasure-burst/minimum delay design criteria forces a unique set of generators for BINARY codes, as per '63 I&C. By '68 I realized these can be obtained from these unitary arrays by reducing all numbers in the arrays mod 2.

The slopes 2/3 & 3/2 correspond to codes of Rates 2/5 & 3/5.

One property I noticed in '63 is that density of ones is low (i.e. odd numbers become sparse). In the conventional wisdom of the late 60s (e.g. Viterbi, Forney, et al) that was viewed as a big disadvantage. But in the past decades we've seen great popularity of very low density parity-check codes AND of turbo codes, both of which are low density (and hard to design in any highly structured way, compared with, say, RS block codes). The key is to find a criteria other than constraint length. Anyway, I think the topic of structured generator sequences for convolutional codes merits attention again, although my '63 paper seems to be perhaps the only one that attempts to initiate work in that area.

I collected lots of examples and techniques circa 1962. That sat in my filing cabinet for about 20 years. I dug them out and wrote up some of them for Sol Golomb's 60th B-Day conference in 1992. Proceedings eventually were published in 2000. I don't have the detailed ref in my head, nor does Sol, nor does Thor Helleson (the editor!). But I'll phone home & get it tomorrow & give it to you then. Track me down!