New Bounds for Multiple Packings of Euclidean Sphere

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Using linear programming lower bounds for the spectrum of code on the Euclidean sphere we obtain new, best known upper bounds for the rate of multiple packing.

Introduction

Let $\mathbb{R}^n$, $\mathbb{S}^{n-1}(\bar{x}, r) \subset \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ be the $n-$dimensional Euclidean space and sphere of radius r with the center in $\bar{x}$. Denote $\mathbb{S}^{n-1} \triangleq \mathbb{S}^{n-1}(\bar{0}, 1)$. Let $\mathbb{B}^n(\bar{x}, r) \subset \mathbb{R}^n$ be the (closed) ball of radius r with the center in $\bar{x}$. We say that (finite) set $\mathcal{K}^n \subset \mathbb{S}^{n-1}(0, r)$ is packing by the balls of radius t of multiplicity L iff for the arbitrary set of

$L+1$ different points $\{\bar{x}_1, \dots, \bar{x}_{L+1}\} \subset \mathcal{K}^n$ we have

$$\bigcap_{i=1}^{L+1} B^n(\bar{x}_i, t) = \emptyset.$$

The same condition can be written as

$$\max_{\bar{x} \in \mathbb{R}^n} \left| B^n(\bar{x}, t) \cap \mathcal{K}^n \right| \le L.$$

Let $t_L(\mathcal{K}^n)$ is the supremum of t with the above property. Next we assume that $n \to \infty$ and $\ln |\mathcal{K}^n| \sim Rn$ for $R > 0$ and we consider the codes on the unit sphere. We are going to obtain the asymptotic upper bound for $t_L = \limsup_{n\to\infty} \sup_{|\mathcal{K}^n| \ge \exp(Rn)} \frac{\sup t_L(\mathcal{K}^n)}{n}$. Next we formulate the previous known bound.

Theorem 1 *The following relation is valid*

$$R \leq \frac{1}{2} \ln \frac{L}{(L+1)t_L^2}. \qquad (1)$$

$$R \geq \frac{1}{2} \ln \frac{L}{(L+1)t_L^2} + \frac{1}{2L} \ln \frac{1}{(L+1)(1-t_L^2)}.$$

We are going to prove the following upper bound on t_L.

Theorem 2 *The following inequality is valid*

$$t_L \leq \left(\frac{1-L}{L} + \frac{1}{\min_{\phi \in (e^{-R},1)} \left(\phi \sin \frac{\theta_\ell (R + \ln \phi)}{2} \right)^2} \right)^{-1/2},$$

where θ_ℓ is the unique root of the equation

$$\frac{1+\sin z}{2\sin z} H\left(\frac{1-\sin z}{1+\sin z}\right) = R \qquad (2)$$

and

$$H(x) = -x\ln x - (1-x)\ln(1-x).$$

As we will see this bound improve bound (1) at low rates.

II Proof of Theorem 2

Denote ($\theta \le 2\varphi$)

$$\begin{aligned}
\alpha_\varphi(\theta) &= 2\arcsin\frac{\sin(\theta/2)}{\sin\varphi}, \\
\beta_\varphi(\theta) &= \arccos\frac{\cos\varphi}{\cos(\theta/2)}, \\
j(x,y) &= (1+y)H\left(\frac{y}{1+y}\right) \\
- & \ln\left(\frac{1}{2}\left(x+\sqrt{(1+2y)^2x^2-4y(1+y)}\right)\right) \\
+ & (1+2y)\times \\
\times & \ln\frac{(1+2y)x+\sqrt{(1+2y)^2x^2-4y(1+y)}}{2y(1+y)}.
\end{aligned}$$

For given R denote by ρ_ℓ the unique solution of the equation

$$R = (1+\rho)H\left(\frac{\rho}{1+\rho}\right).$$

Denote

$$
\begin{aligned}
& b_{\mathcal{K}}(x, \bar{c}) \\
= \ & \{\bar{c}_1 : \ (\bar{c}, \bar{c}_1) \in \mathcal{K} \times \mathcal{K}, \ (\bar{c}, \bar{c}_1)/(||\bar{c}||||\bar{c}_1||) = x\},
\end{aligned}
$$

where $(\bar{a}, \bar{b}) = a_1 b_1 + \ldots a_n b_n$ stands for the scalar product. We will use the following result.

Theorem 3 *For* $\mathcal{K}^n \subset \mathbb{S}^{n-1}(\bar{0}, r)$, *where* $|\mathcal{K}^n| = Rn(1 + o(n))$ *and* ρ, φ *satisfying*

$$
e^{-R} \leq \sin \varphi \leq 1, \ 0 \leq \rho \leq \rho_\ell(R + \ln \sin \varphi)
$$

there exists θ *and* $\bar{c} \in \mathcal{K}^n$ *such that*

$$
\frac{2\sqrt{\rho(1+\rho)}}{1+2\rho} \leq \cos \alpha_\varphi(\theta) \qquad (3)
$$

and

$$\frac{1}{n}\ln b_{\mathcal{K}^n}(\cos\theta,\overline{c}) \geq R + 2\ln\sin\varphi - \ln\sin\beta\varphi(\theta)$$
$$- \quad j(\cos\alpha_{\varphi}(\theta),\rho) + o(1).$$

It is easy to see that $b_{\mathcal{K}^n}(\cos\theta,\overline{c})$ is the number of code vectors in the cross-section of the sphere by the hyperplane orthogonal to the vector $\overline{c}$ in the point $r\cos\theta\frac{\overline{c}}{||\overline{c}||}$, where r is the radius of the sphere.

Next we describe a recursive procedure of constructing a simplex of $L+1$ codewords $\{\overline{c}_0,\ldots,\overline{c}_L\}\subset\mathcal{K}^n$ which has sufficiently small pairwise distances and thus sufficiently small minimal radius r of the ball $B^n(\overline{x},r)$ which contains all points from the simplex.

We start with the code $\mathcal{K}_0^n \subset \mathbb{S}^{n-1}$ and consider the cross-section of $\mathbb{S}^{n-1}$ by the hyperplane. Then for some $\bar{c}_0$ the following is true: for each φ_1, ρ_1 as in the Theorem 3 there exists θ_1, as in the Theorem 3, such that the cross-section of $\mathbb{S}^{n-1}$ by the hyperplane orthogonal to $\bar{c}_0$ and having distance $\cos\theta_1$ from the origin, contains the number of code vectors $b_{\mathcal{K}_0^n}(\cos\theta_1, \bar{c}_0)$.

Next note that the cross- section of the sphere by hyperplane is again the sphere of the dimension one less. New sphere has radius $r \sin\theta$ and center in $r\cos(\theta)\frac{\bar{c}_0}{||\bar{c}_0||}$ (if the initial sphere has radius r.)

Next we shift the center of this new sphere to the origin and once again consider the cross- section of this new sphere by hyperplane as in the previous step. Now we have the new code $\mathcal{K}_1^n$ of cardinality $b_{\mathcal{K}_0^n}(\cos\theta_1, \bar{c}_0)$ and the new cross- section contains for given $\arcsin(e^{-\ln b_{\mathcal{K}_0^n}(\cos\theta_1, \bar{c}_0)/n}) \leq \varphi_2 \leq \pi/2,\ 0 \leq \rho_2 \leq \rho_\ell(\ln b_{\mathcal{K}_0^n}(\cos\theta_1, \bar{c})/n + \ln\sin\varphi_1)$ at least

$$b_{\mathcal{K}_1^n}(\cos\theta_2, \rho_1)$$

code vectors from $\mathcal{K}_1^n$, where

$$\begin{aligned} \frac{\ln b_{\mathcal{K}_1^n}(cos\theta_2, \bar{c}_1)}{n} &\geq \frac{\ln b_{\mathcal{K}_0^n}(cos\theta_1, \bar{c}_0)}{n} + 2\ln\sin\varphi_2 \\ - &\quad \ln\sin\beta_{\varphi_2}(\theta_2) \\ - &\quad j(\cos\alpha_{\varphi_2}(\theta_2), \rho_2) + o(1). \end{aligned}$$

At this, second step we choose new $\varphi_2, \rho_2, \theta_2, \bar{c}_1$.

This procedure can be continued. Let us provide a formal description of the procedure. On $0-$th step we have code $\mathcal{K}_0^n \subset \mathbb{S}^{n-1}$, $\ln|\mathcal{K}_0^n| \sim nR_0$ $(R_0 = R)$. On $i-$th step, $i \geq 1$ we obtain a code $\mathcal{K}_i^n$ of the rate $\frac{1}{n}\ln|\mathcal{K}_i^n| \sim R_i$ such that

$$R_i \geq R_{i-1} + 2\ln\sin\varphi_i - \ln\sin\beta_{\varphi_i}(\theta_i) - j(\cos\alpha_{\varphi_i}(\theta_i), \rho_i).$$

We implement this action $L+1$ times, and on the $i-$th step, $i \geq 1$ we fix a new code vector $\bar{c}_i$ such that its distance from $\bar{c}_j$, $0 \leq j < i$ is $d_j = 2r_j \sin\frac{\theta_{j+1}}{2}$. We stop when we fix $L+1$ code vectors $\bar{c}_i \in \mathcal{K}_i^n$, $i = 0, 1, \ldots, L$. Note also that $\mathcal{K}_L^n \subset \mathcal{K}_{L-1}^n \subset \ldots \subset \mathcal{K}_0^n$. What should be done next is to optimize the set d_j, $j = 0, 1, \ldots, L-1$ in such a way that the simplex on the vertices

$\bar{c}_i$ is contained in the ball of minimal possible radius t_L (we should optimize over φ_i, ρ_i and at the same time consider the worst case for the choice of θ_i).

On this way we state the existence of the point in $\mathbb{R}^n$ which is covered by $L+1$ balls of radius t_L with the centers in coding points, which yield that an arbitrary code of rate R on the Euclidean sphere is $L-$packing by the ball of radius strictly less than t_L.

We call the set $(\theta_1, \dots, \theta_L)$ admissible if there exist sets $\varphi_1, \dots, \varphi_L$; $\rho_1, \dots, \rho_L$ such that $e^{-R_{i-1}} \le \sin \varphi_i$, $0 \le \rho_i \le \rho_\ell(R_{i-1} + \ln \sin \varphi_{i-1})$ and

$$R_i = R_{i-1} + 2 \ln \sin \varphi_i - \ln \sin \beta_{\varphi_i}(\theta_i) - j(\cos \alpha_{\varphi_i}(\theta_i), \rho_i),$$

We say that the set $(\varphi_1, \dots, \varphi_L,\ \rho_1, \dots, \rho_L)$ is admissible if there exists set $\theta_1, \dots, \theta_L$ which is admissible for these φ_i, ρ_i .

If $t(d_0, \dots, d_{L-1})$ is the minimal radius of the ball which contains the simplex $\{\bar{c}_0, \dots, \bar{c}_L\}$ such that $||\bar{c}_i - \bar{c}_j|| = d_i = 2r_i \sin\frac{\theta_{i+1}}{2},\ i < j$ and $r_i = r_{i-1}\sin\theta_i,\ r_0 = 1$ then

$$t(d_0, \dots, d_{L-1}) = \max_{I : \gamma^I(\Gamma^I)^{-1}(\gamma^I)^T \geq 0} \frac{1}{2}\sqrt{\gamma^I(\Gamma^I)^{-1}(\gamma^I)^T}, \tag{4}$$

where $\gamma = (\gamma_1, \dots, \gamma_L),\ \Gamma = ||\Gamma_{ij}||$ and

$$\gamma_i = d_0^2,\ \Gamma_{i,j} = (\bar{c}_0 - \bar{c}_i, \bar{c}_0 - \bar{c}_j)$$

and γ^I, Γ^I are vector and matrix which are obtained from the initial vector γ and matrix Γ by deleting elements whose indices i

and i, j do not belong to I. Also note that formula (4) have sense if Γ is not singular, i.e. (because it is Gramm matrix), vectors $\bar{c}_i - \bar{c}_0$ are linear independent. As we will see it is our case.

Then we have the upper bound

$$t_L \leq \tag{5}$$

$$\leq \max_{admiss\ \{\varphi_i,\ \rho_i\}} \min_{admiss\{\theta_i\}\ for\ \{\varphi_i, \rho_i\}} t(d_0, \ldots, d_{L-1}). \tag{6}$$

One can see that bound (5) is rather complicated, used difficult optimizations, and it is not written in the closed analytic form. Next we simplify this bound. Possibly this worsen bound (5) but allow us to demonstrate on this way we improve bound (1) for low rates. We bring our attention to the case when $d_i = d_j, i, j > 1$ and moreover to the case when on the first step we choose the angle θ_1 which can be found from the equation

$$R = R_0 = \min_{\theta/2 \le \varphi \le \pi/2} (R_\ell(\alpha_\varphi(\theta)) - \ln \sin \varphi) \tag{7}$$

or

$$\begin{aligned} & \theta_1 = \theta'_\ell(R) \\ \triangleq\ & 2 \min_{e^{-R} \le \phi \le \pi/2} \arcsin \left(\phi \sin \frac{\theta_\ell(R + \ln \sin \phi)}{2} \right). \end{aligned}$$

Let's start from the code $\mathcal{K}_0 \subset \mathbb{S}^{n-1}$ of rate $R = R_0$. Let $\varphi_1 = \varphi$–be the value on which minimum in (7) is achieved. Set $\rho_1 =$

$\rho_\ell(R+\ln\sin\varphi_0)-\epsilon,\ \ \epsilon << \rho_\ell(R+\ln\sin\varphi)$.
The function

$$f(\theta,\rho) = -2\ln\sin\varphi + \ln\sin\beta_\varphi(\theta) + j(\cos\alpha_\varphi(\theta),\rho)$$

is increasing in θ and

$$f(0,\rho) = (1+\rho)H\left(\frac{\rho}{1+\rho}\right).$$

Thus for some $\theta_1 < \theta'_\ell(R)+\delta,\ \ \delta << \theta'_\ell$,

$$R_1 \geq R-(R+\ln\sin\varphi)-f(\theta_1,\rho_1) = \epsilon_1.$$

Next for the code $\mathcal{K}_1^n \subset \mathbb{S}^{n-1}(\bar{0},\sin\theta'_\ell(R))$ of rate R_1 we use Plotkin bound which is the particular case of (1). Actually in [**?**] was proved that for fixed N from the arbitrary code (on the sphere of fixed radius) of growing size one can extract simplex

$\bar{c}_1, \ldots, \bar{c}_N$ such that

$$d(\bar{c}_i, \bar{c}_j) = d + o(1),\ i \neq j$$

i.e. pairwise distances between vectors from this simples asymptotically do not depend of choice of these pairs. Then we have $||\bar{c}_i - \bar{c}_j|| \leq \sqrt{2}r$, where r is radius of the sphere and in our case

$$r = \sin \theta'_\ell(R)$$

and we have

$$||\bar{c}_i - \bar{c}_j|| \leq \sqrt{2} \sin \theta'_\ell(R).$$

Note that this last considerations consistent with the previous procedure where we consequently choose codes $\mathcal{K}_1^n, \ldots, \mathcal{K}_L^n$. Indeed on each step $i \geq 1$ we choose $\mathcal{K}_i^n$, such that $|\mathcal{K}_i^n| = exp(\epsilon_i n)$ with minimal distance $d \leq \sqrt{2}\sin\theta'_\ell(R)$.

Now we have simplex $\bar{c}_0, \ldots \bar{c}_L$ with $d(\bar{c}_i, \bar{c}_j) = d + o(1),\ \ i \neq j > 0$. As we have already mention the minimal radius r_L of closed ball which contains all these points is determinate by the equality (4).

From cosines theorem it follows that

$$\begin{aligned} & d^2 + o(1) = (\bar{c}_i - \bar{c}_j)^2 \\ = \ & (\bar{c}_i - \bar{c}_0)^2 + (\bar{c}_j - \bar{c}_0)^2 - 2(\bar{c}_i - \bar{c}_0, \bar{c}_j - \bar{c}_0) \\ = \ & 2r^2 - 2(\bar{c}_i - \bar{c}_0, \bar{c}_j - \bar{c}_0) \end{aligned}$$

and (up to $o(1)$)

$$\Gamma_{ij} = r^2 - d^2/2,\ i \neq j$$

and

$$\Gamma_{ii} = r^2.$$

Here r is radius of the sphere which is cross-section of the unit sphere by the hyperplane.

Thus we have

$$\Gamma = \left(\begin{array}{cccccc} a & b & b & b & \dots & b \\ b & a & b & b & \dots & b \\ \vdots & & & & \dots & \\ b & b & b & b & \dots & a \end{array} \right),$$

where

$$a = r^2,\ b = r^2 - d^2/2 \qquad (8)$$

and Γ^I has the same form and $\gamma = (a, a, \dots, a)$. We have

$$\det \Gamma^I = (a-b)^{|I|-1}((|I|-1)b + a).$$

At the same time

$$\det \Gamma^I = aA^I_{ii} + b \sum_{k:\ k\neq i} A^I_{ik}.$$

$\gamma^I(\Gamma^I)^{-1}(\gamma^I)^T > 0$ for all nonempty I. and max in (4) achieves on $I = \{1, 2, \dots L\}$. we have

$$\gamma\Gamma^{-1}\gamma^T = \frac{a^2 L}{(L-1)b + a}$$

and

$$t = \frac{a}{2}\sqrt{\frac{L}{(L-1)b + a}}.$$

Taking into account relation (8) we obtain the equality

$$t = \frac{r^2}{2}\sqrt{\frac{L}{(L-1)(r^2 - d^2/2) + r^2}}$$
$$= \frac{r}{2}\left(1 - \frac{L-1}{2L}\frac{d^2}{r^2}\right)^{-1/2} .$$

Because $r = 2\sin\frac{\theta'_\ell(R)}{2}$ and (we omit $o(1)$) $d = \sqrt{2}\sin\theta'_\ell(R) = 2\sqrt{2}\sin\frac{\theta'_\ell(R)}{2}\cos\frac{\theta'_\ell(R)}{2}$ from (**??**) we have

$$t = \frac{\sin\frac{\theta'_\ell(R)}{2}}{\sqrt{\frac{1}{L}\cos^2\frac{\theta'_\ell(R)}{2} + \sin^2\frac{\theta'_\ell(R)}{2}}}$$
$$= \left(1 + \frac{1}{L}\cot^2\frac{\theta'_\ell(R)}{2}\right)^{-1/2}$$

or

$$\cot\frac{\theta'_\ell(R)}{2} = \sqrt{Lt^{-2}-L},$$

$$\theta'_\ell(R) = 2arccot\sqrt{Lt^{-2}-L}.$$

Now using (7) we obtain bound

$$\begin{aligned} & 2arccot\sqrt{Lt^{-2}-L} \\ \leq\ & 2\min_{\arcsin e^{-R}\leq\varphi\leq\pi/2}\arcsin\left(\sin\varphi\sin\frac{\theta_\ell(R+\ln\sin\varphi)}{2}\right. \end{aligned}$$

Denote $q = 1/\min_{e^{-R}\leq\phi\leq 1}\left(\phi\sin\frac{\theta_\ell(R+\ln\phi)}{2}\right)$.
Then (**??**) is equivalent to the inequality

$$t_L \leq \left(\frac{L-1}{L}+q^2/L\right)^{-1/2}. \qquad (9)$$

This is our final bound. Theorem 2 is proved. On the Fig.1 are drawn graph of these bounds for $L = 1, 2, 5$ and graphs of the bound

$$t_L \leq \sqrt{\frac{L}{L+1}} e^{-R}, \qquad (10)$$

which is equivalent to (1).

One can easily see that derivative over R of the rhs of (9) is

$$\frac{2qq'_R}{(q^2/L + (L-1)/L)^{3/2}} \overset{R \to 0}{\longrightarrow} -\infty$$

and at the same time derivative of he rhs of (10) at zero is equal to $-\sqrt{\frac{L}{L+1}}$ and they both are equal $\sqrt{\frac{L}{L+1}}$ a zero rate. This proves that bound (9) is better at low rates.

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