A Note on a Result by Hamada on Minihypers

Ivan Landjev

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Assia Rousseva

Faculty of Mathematics and Informatics
Sofia University

– XI International Workshop on ACCT, Pamporovo, June 16–22, 2008 –
Linear Codes and the Griesmer Bound

$C : \text{linear } [n, k, d]-\text{code over } \mathbb{F}_q$

The Griesmer bound:

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \overset{\text{def}}{=} g_q(k, d)$$

Griesmer code: $[g_q(k, d), k, d]_q$. 
Multisets and Minihypers in $\text{PG}(t, q)$

$\text{PG}(t, q) = (\mathcal{P}, \mathcal{L}, I)$

$v_{t+1} = \frac{q^{t+1} - 1}{q-1}$

A multiset in $\text{PG}(t, q)$: $\mathcal{K} : \mathcal{P} \rightarrow \mathbb{N}_0$.

The support of $\mathcal{K}$: $\text{Supp } \mathcal{K} = \{P \in \mathcal{P} \mid \mathcal{K}(P) > 0\}$.

A projective multiset in $\text{PG}(t, q)$: $\text{Im } \mathcal{K} = \{0, 1\}$.

A characteristic multiset of a set $Q \subseteq \mathcal{P}$:

$$\chi_Q = \begin{cases} 
1 & P \in Q, \\
0 & Q \notin Q.
\end{cases}$$
Multisets and Minihypers in $\text{PG}(t, q)$ (cont.)

**Definition.** A multiset $\mathcal{F}$ in $\text{PG}(t, q)$, $t \geq 2$, is called an $(f, m; t, q)$-minihyper or $(f, m)$-minihyper if

(a) $\mathcal{F}(P) = f$;

(b) $\mathcal{F}(H) \geq m$ for any hyperplane $H$;

(c) there exists a hyperplane $H_0$ with $\mathcal{F}(H_0) = m$.

An $(f, m; t, q)$-minihyper $\mathcal{F}$ is said to be reducible if there exists an $(f', m; t, q)$-minihyper $\mathcal{F}'$ with $f' < f$ and $\mathcal{F}'(P) \leq \mathcal{F}(P)$ for every $P \in \mathcal{P}$. 
Minihypers and Linear Codes

Let $C$ be an $[n, k, d]_q$-code.

If $(s - 1)q^{k-1} \leq d < sq^{k-1}$ then $d$ can be written (uniquely) as:

$$d = sq^{k-1} - \sum_{i=1}^{h} q^{\lambda_i},$$

where

(a) $0 \leq \lambda_1 \leq \ldots \leq \lambda_h < k - 1$;

(b) at most $q - 1$ of the values $\lambda_i$ are equal to a given value.

The Griesmer bound:

$$n \geq g_q(k, d) = sv_k - \sum_{i=1}^{h} v_{\lambda_i + 1}$$
Minihypers and Linear Codes (cont.)

**Theorem.** *(Hamada)*

If \( d = sq^{k-1} - \sum_{i=1}^{h} q^{\lambda_i} \) there exists a one-to-one correspondence between the set of nonequivalent \([n, k, d]_q\) codes meeting the Griesmer bound and the set of all projectively distinct minihypers with parameters

\[
\left( \sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k - 1, q \right).
\]
Characterization Results on Minihypers

**Theorem.** *(Hamada, Helleseth, Maekawa)*

A non-weighted minihyper with parameters

\[
\left( \sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; t, q \right),
\]

where \( \sum_{i=0}^{t-1} \epsilon_i < \sqrt{q} + 1 \), is a union of \( \epsilon_{t-1} \) hyperplanes, \( \epsilon_{t-2} \) \((t - 2)\)-dimensional spaces, \( \ldots \), \( \epsilon_1 \) lines, and \( \epsilon_0 \) points, which all are pairwise disjoint.
Characterization Results on Minihypers (cont.)

Theorem. (De Beule, Metsch, Storme)

A \( \left( \sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; t, q \right) \)-minihyper, where \( \sum_{i=0}^{t-1} \epsilon_i \leq \delta_0 \), with \( \delta_0 \) equal to one of the values in the table below, is a union of \( \epsilon_{t-1} \) hyperplanes, \( \epsilon_{t-2} \) \((t - 2)\)-dimensional spaces, \ldots, \( \epsilon_1 \) lines, and \( \epsilon_0 \) points, which all are pairwise disjoint. (In the table \( q = p^s \), \( p \) prime, \( s \geq 1 \).)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( s )</th>
<th>( \delta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>even</td>
<td>( \leq \sqrt{q} )</td>
</tr>
<tr>
<td>( p )</td>
<td>( s = 1 )</td>
<td>( \leq (p + 1)/2 )</td>
</tr>
<tr>
<td>( p )</td>
<td>3</td>
<td>( \leq p^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 6m + 1, m \geq 1 )</td>
<td>( \leq 2^{4m+1} - 2^{4m} - 2^{2m+1}/2 )</td>
</tr>
<tr>
<td>( &gt; 2 )</td>
<td>( 6m + 1, m \geq 1 )</td>
<td>( \leq p^{4m+1} - p^{4m} - p^{2m+1}/2 + 1/2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 6m + 3, m \geq 1 )</td>
<td>( &lt; 2^{4m+5/2} - 2^{4m+1} - 2^{2m+1} + 1 )</td>
</tr>
<tr>
<td>( &gt; 2 )</td>
<td>( 6m + 3, m \geq 1 )</td>
<td>( \leq p^{4m+2} - p^{2m+2} + 2 )</td>
</tr>
<tr>
<td>( \geq 5 )</td>
<td>( 6m + 5, m \geq 0 )</td>
<td>( &lt; p^{4m+7/2} - p^{4m+3} - p^{2m+2}/2 + 1 )</td>
</tr>
</tbody>
</table>

Table 1: upper bounds on \( \delta_0 \)
Characterization Results on Minihypers (cont.)

**Theorem.** *(Hamada)* A non-weighted

\[
\left( \sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q \right)
\]

minihyper, with \( t > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0 \), is the union of a \( \lambda_1 \)-dimensional space, \( \lambda_2 \)-dimensional space, \( \ldots \), \( \lambda_h \)-dimensional space, which all are pairwise disjoint.
Characterization Results on Minihypers (cont.)

**Theorem.** (Landjev, Storme)

Let $\mathfrak{F}$ be a $\left( \sum_{i=1}^{h} v_{\lambda_{i}+1}, \sum_{i=1}^{h} v_{\lambda_{i}}; t, q \right)$-minihyper, with $t \geq 2$, $q \geq 3$, and

$$t > \lambda_{1} > \lambda_{2} > \cdots > \lambda_{h} \geq 0.$$

Then

$$\mathfrak{F} = \sum_{i=1}^{h} \chi_{\pi_{i}},$$

where $\pi_{i}$ is a $\lambda_{i}$-dimensional subspace of $\text{PG}(t, q)$ and $i = 1, \ldots, h.$
The Theorem

**Theorem.** Let \( t \geq 2 \) be an integer and let \( q \geq 3 \) be a prime power. Let \( \lambda_1, \ldots, \lambda_h \) be a sequence of non-negative integers such that

(1) \( t > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_h \geq 0 \), and

(2) equalities in (1) occur in at most \( r(q) - 1 \) places, where \( q + 1 + r(q) \) is the size of the smallest nontrivial blocking set in \( \text{PG}(2, q) \).

Then every minihyper \( \mathcal{F} \) in \( \text{PG}(t, q) \) with parameters \((\sum_{i=1}^{h} v\lambda_{i+1}, \sum_{i=1}^{h} v\lambda_i)\) can be represented as

\[
\mathcal{F} = \sum_{i=1}^{h} \chi_{\pi_i},
\]

where \( \pi_i \) is a \( \lambda_i \)-dimensional subspace of \( \text{PG}(t, q) \) and \( i = 1, \ldots, h \).