A Note on a Result by Hamada on Minihypers

Ivan Landjev

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Assia Rousseva

Faculty of Mathematics and Informatics
Sofia University

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Linear Codes and the Griesmer Bound

C : linear $[n, k, d]$-code over $\mathbb{F}_q$

The Griesmer bound:

$$n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil \stackrel{def}{=} g_q(k, d)$$

Griesmer code: $[g_q(k, d), k, d]_q$.

Multisets and Minihypers in $\mathrm{PG}(t,q)$

$\mathrm{PG}(t,q) = (\mathcal{P}, \mathcal{L}, I)$

$v_{t+1} = \frac{q^{t+1}-1}{q-1}$

A multiset in $\mathrm{PG}(t,q)$: $\mathfrak{K}$: $\mathcal{P} \to \mathbb{N}_0$.

The support of $\mathfrak{K}$: $\operatorname{Supp}\mathfrak{K} = \{P \in \mathcal{P} \mid \mathfrak{K}(P) > 0\}$.

A projective multiset in $\mathrm{PG}(t,q)$: $\operatorname{Im}\mathfrak{K} = \{0,1\}$.

A characteristic multiset of a set $\mathcal{Q} \subseteq \mathcal{P}$:

$$\chi_{\mathcal{Q}} = \begin{cases} 1 & P \in \mathcal{Q}, \\ 0 & Q \notin \mathcal{Q}. \end{cases}$$

Multisets and Minihypers in $\mathrm{PG}(t,q)$ (cont.)

Definition. *A multiset $\mathfrak{F}$ in $\mathrm{PG}(t,q)$, $t \geq 2$, is called an $(f,m;t,q)$-*minihyper *or (f,m)-minihyper if*

(a) $\mathfrak{F}(\mathcal{P}) = f$;

(b) $\mathfrak{F}(H) \geq m$ *for any hyperplane* H;

(c) there exists a hyperplane H_0 with $\mathfrak{F}(H_0) = m$.

An $(f,m;t,q)$-minihyper $\mathfrak{F}$ is said to be *reducible* if there exists an $(f',m;t,q)$-minihyper $\mathfrak{F}'$ with $f' < f$ $\mathfrak{F}'(P) \leq \mathfrak{F}(P)$ for every $P \in \mathcal{P}$.

Minihypers and Linear Codes

Let C be an $[n, k, d]_q$-code.

If $(s-1)q^{k-1} \leq d < sq^{k-1}$ then d can be written (uniquely) as:

$$d = sq^{k-1} - \sum_{i=1}^{h} q^{\lambda_i},$$

where

(a) $0 \leq \lambda_1 \leq \ldots \leq \lambda_h < k-1$;

(b) at most $q-1$ of the values λ_i are equal to a given value.

The Griesmer bound:

$$\boxed{n \geq g_q(k, d) = sv_k - \sum_{i=1}^{h} v_{\lambda_i+1}}$$

Minihypers and Linear Codes(cont.)

Theorem. *(Hamada)*

If $d = sq^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$ *there exits a one-to-one correspondence between the set of nonequivalent* $[n, k, d]_q$ *codes meeting the Griesmer bound and the set of all projectively distinct minihypers with parameters*

$$(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k-1, q).$$

Characterization Results on Minihypers

Theorem. *(Hamada, Helleseth, Maekawa)*

A non-weighted minihyper with parameters

$$\left(\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; t, q\right),$$

where $\sum_{i=0}^{t-1} \epsilon_i < \sqrt{q}+1$, *is a union of* ϵ_{t-1} *hyperplanes,* ϵ_{t-2} $(t-2)$*-dimensional spaces,* $\ldots, \epsilon_1$ *lines, and* ϵ_0 *points, which all are pairwise disjoint.*

Characterization Results on Minihypers (cont.)

Theorem. *(De Beule, Metsch, Storme)*

A $(\sum_{i=0}^{t-1} \epsilon_i v_{i+1}, \sum_{i=0}^{t-1} \epsilon_i v_i; t, q)$*-minihyper, where* $\sum_{i=0}^{t-1} \epsilon_i \leq \delta_0$*, with* δ_0 *equal to one of the values in the table below, is a union of* ϵ_{t-1} *hyperplanes,* ϵ_{t-2} $(t-2)$*-dimensional spaces,* $\ldots, \epsilon_1$ *lines, and* ϵ_0 *points, which all are pairwise disjoint. (In the table* $q = p^s$*,* p *prime,* $s \geq 1$*.)*

p	s	δ_0
p	even	$\leq \sqrt{q}$
p	$s = 1$	$\leq (p+1)/2$
p	3	$\leq p^2$
2	$6m+1, m \geq 1$	$\leq 2^{4m+1} - 2^{4m} - 2^{2m+1}/2$
> 2	$6m+1, m \geq 1$	$\leq p^{4m+1} - p^{4m} - p^{2m+1}/2 + 1/2$
2	$6m+3, m \geq 1$	$< 2^{4m+5/2} - 2^{4m+1} - 2^{2m+1} + 1$
> 2	$6m+3, m \geq 1$	$\leq p^{4m+2} - p^{2m+2} + 2$
≥ 5	$6m+5, m \geq 0$	$< p^{4m+7/2} - p^{4m+3} - p^{2m+2}/2 + 1$

Table 1: upper bounds on δ_0

Characterization Results on Minihypers (cont.)

Theorem. *(Hamada) A non-weighted*

$$\left(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q\right)$$

minihyper, with $t > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0$*, is the union of a* λ_1*-dimensional space,* λ_2*-dimensional space, ...,* λ_h*-dimensional space, which all are pairwise disjoint.*

Characterization Results on Minihypers (cont.)

Theorem. *(Landjev, Storme)*

Let $\mathfrak{F}$ *be a* $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; t, q)$*-minihyper, with* $t \geq 2$, $q \geq 3$, *and*

$$t > \lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0.$$

Then

$$\mathfrak{F} = \sum_{i=1}^{h} \chi_{\pi_i},$$

where π_i *is a* λ_i*-dimensional subspace of* $\mathrm{PG}(t, q)$ *and* $i = 1, \ldots, h$.

The Theorem

Theorem. *Let* $t \geq 2$ *be an integer and let* $q \geq 3$ *be a prime power. Let* $\lambda_1, \ldots, \lambda_h$ *be a sequence of non-negative integers such that*

(1) $t > \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_h \geq 0$*, and*

(2) equalities in (1) occur in at most $r(q) - 1$ *places, where* $q+1+r(q)$ *is the size of the smallest nontrivial blocking set in* $\mathrm{PG}(2, q)$*.*

Then every minihyper $\mathfrak{F}$ *in* $\mathrm{PG}(t, q)$ *with parameters* $(\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i})$ *can be represented as*

$$\mathfrak{F} = \sum_{i=1}^{h} \chi_{\pi_i},$$

where π_i *is a* λ_i*-dimensional subspace of* $\mathrm{PG}(t, q)$ *and* $i = 1, \ldots, h$*.*