Symmetric configurations for bipartite-graph codes

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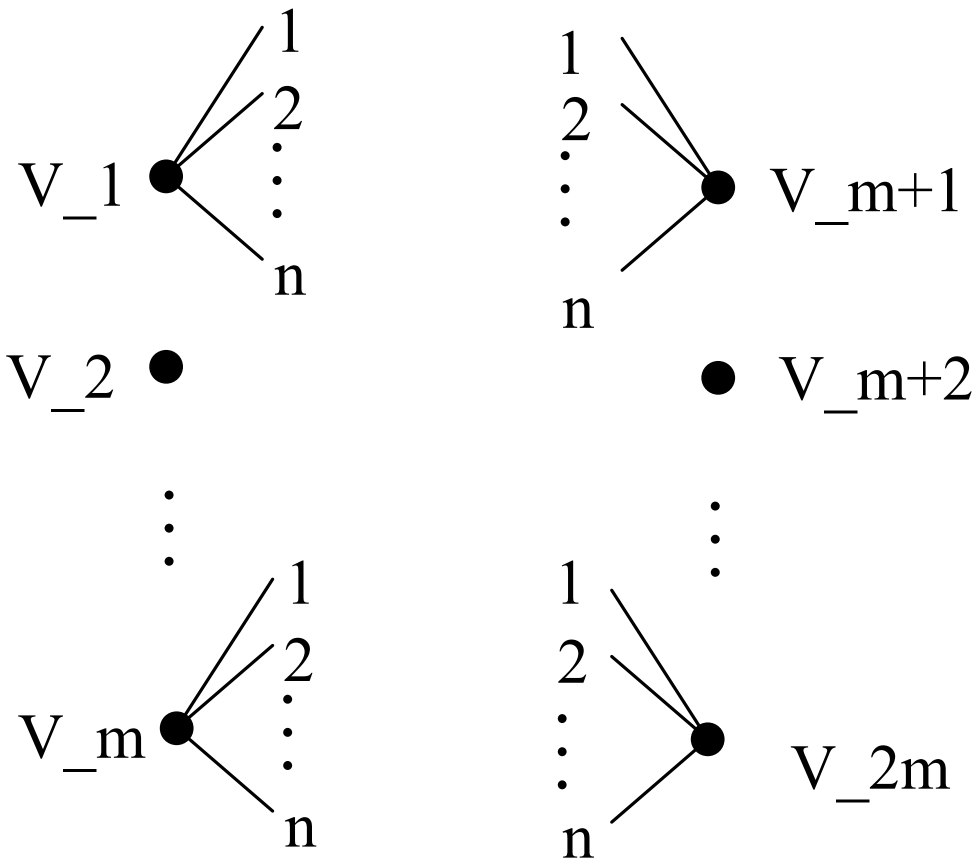
Supporting graph of BG code



Bipartite-graph code (BG code)

G - an n-regular bipartite graph (*supporting graph*)

Classes of vertices: $\{V_1, \ldots, V_m\}, \quad \{V_{m+1}, \ldots, V_{2m}\}$

$\mathcal{C}_t$ - an $[n, k_t]$ *constituent code*, $t = 1, 2, \ldots, 2m$

Bipartite-graph code $\mathcal{C}$ - an $[N, K]$ code. $\quad N = mn$

Positions of a **codeword** of $\mathcal{C} \Leftrightarrow$ **edges** of G

constituent code $\mathcal{C}_t \Leftrightarrow$ **vertex** V_t :

projection of a codeword of $\mathcal{C}$ to the positions noted by n edges incident to a vertex $V_t \quad = \quad$ codeword of $\mathcal{C}_t$

graph $G \Leftrightarrow$ 01-matrix $M(m,n)$

$M(m,n)$ - square 01-matrix of order m with n units in every row and column

i-th row (j-th column) $\Leftrightarrow$ vertex V_i (vertex V_{m+j})

"1" in position $(i,j) \Leftrightarrow V_i$ and V_{m+j} are adjacent

$$\text{critical submatrix } J_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

graph G is 4-cycle free $\Leftrightarrow$ matrix $M(m,n)$ is J_4-free

GOAL: *to construct J_4-free matrices $M(m,n)$ with* ***distinct parameters*** $m,n \Leftrightarrow$
to increase the region of lengths and rates of BG codes

Designs and configurations

symmetric and resolvable non-symmetric
2-$(v, k, 1)$ design (= Steiner system $S(2, k, v)$) :

every pair of elements lies in *exactly* one block

2-$(v, k, 1)$ **design** $\Rightarrow$ J_4-free matrix $M(v, k)$

Configuration (v_r, b_k) $\Rightarrow$ v elements, b blocks
every block contains k points
every element belongs to r blocks
every pair of elements belongs to *at most* one block

Symmetric configuration : $v = b, \ r = k \quad \Rightarrow$
J_4-free matrix $M(v, k)$

Construction A in $PG(s,q)$

a single orbit of a collineation group in $PG(s,q)$

$\mathcal{P}$ - a **point orbit** under the action of a collineation group in $PG(s,q)$ or $AG(s,q)$

$\mathcal{L}(\mathcal{P},n)$ - the set of lines meeting $\mathcal{P}$ in n points

incidence structure: *points* = points of $\mathcal{P}$
lines = lines of $\mathcal{L}(\mathcal{P},n)$
incidence = incidence of the starting space

$M(m,n) \Leftrightarrow$ **incidence matrix**, $m = |\mathcal{P}|, n \leq q+1$

Construction A works for any 2-$(v,k,1)$ design D and for any group of automorphisms of D

Theorem and Examples

Theorem. In Construction A the number of lines of $\mathcal{L}(\mathcal{P}, n)$ through a point of $\mathcal{P}$ is a constant r_n

$n = r_n \implies$ J_4-free matrix $M(|\mathcal{P}|, n)$

Examples: $\mathcal{P}$ is the set of internal points to a conic; $\mathcal{L}(\mathcal{P}, n)$ is the set of lines external to the conic.

$$M(m, n) : m = \frac{1}{2}q(q-1),\ n = \frac{1}{2}(q+1),\ q \text{ odd}$$

$\mathcal{P}$ is the complement of a Baer subplane of $PG(2, q)$; $\mathcal{L}(\mathcal{P}, q)$ is the set of tangents to the subplane.

$$M(m, n) : m = q^2 - \sqrt{q} < n^2,\ n = q,\ q \text{ square}$$

orbits of a Singer subgroup $\widehat{S}_d$

Singer group S - the cyclic collineation group of order q^2+q+1 in $PG(2,q)$

d - any divisor of q^2+q+1. $\quad t=\frac{q^2+q+1}{d}$

$\widehat{S}_d$ - the unique cyclic subgroup of S of order d

under the action of a **cyclic** collineation group
the point set and the line set of a projective plane have the same cyclic structure

$O_0, O_1, \ldots, O_{t-1}$ - point orbits under the action of $\widehat{S}_d$

$L_0, L_1, \ldots, L_{t-1}$ - line orbits under the action of $\widehat{S}_d$

Partition of $PG(2,q)$ into orbits

Theorem. Every line of the orbit L_i meets the orbit O_j in the same number of points $w_{j-i \pmod t}$

Incidence matrix of $PG(2,q)$

$$V = \begin{bmatrix} C_{0,0} & C_{0,1} & C_{0,2} & \cdots & C_{0,t-1} \\ C_{1,0} & C_{1,1} & C_{1,2} & \cdots & C_{1,t-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{t-1,0} & C_{t-1,1} & C_{t-1,2} & \cdots & C_{t-1,t-1} \end{bmatrix}$$

$C_{i,j}$ - **circulant** $d \times d$ matrix of weight $w_{j-i \pmod t}$

matrix of weights of submatrices

$$W = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & \dots & w_{t-2} & w_{t-1} \\ w_{t-1} & w_0 & w_1 & w_2 & \dots & w_{t-3} & w_{t-2} \\ w_{t-2} & w_{t-1} & w_0 & w_1 & \dots & w_{t-4} & w_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1 & w_2 & w_3 & w_4 & \dots & w_{t-1} & w_0 \end{bmatrix}$$

Circulant $d \times d$ matrix C of weight w

$$\Downarrow$$

Circulant $d \times d$ matrix C' of weight $w - \delta$

$$V \Rightarrow V' \qquad W \Rightarrow W'$$

an union of orbits of $\widehat{S}_d$

Construction B in $PG(2, q)$

$\frac{m}{d} \times \frac{m}{d}$ submatrix of W' :

sum of elements of every row and every column = n

$$\Downarrow$$

submatrix of V' is a J_4-free matrix $M(m, n)$

game of blocks (bricks for children)

numerous matrices square and non-square

n_1 units in every row; n_2 units in every column

$n_1 = n_2$ and $n_1 \neq n_2$

Parity check matrix

$$\begin{bmatrix} cI_m & cI_m & \ldots & cI_m \\ \vdots & \vdots & \vdots & \vdots \\ cI_m & cI_m & \ldots & cI_m \\ cI_m^{(1)} & cI_m^{(2)} & \ldots & cI_m^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ cI_m^{(1)} & cI_m^{(2)} & \ldots & cI_m^{(n)} \end{bmatrix} . \quad I_m \text{ - identity matrix}$$

$M(m,n)$ - circulant. $I_m^{(v)}$ - circulant permutation $m \times m$ matrix obtained by decomposition of $M(m,n)$

Constants c are distinct!!! They comes from parity check matrices of the constituent codes:
$\mathcal{C}_1 = \ldots = \mathcal{C}_m \qquad \mathcal{C}_{m+1} = \ldots = \mathcal{C}_{2m}$

Thank you

Mille grazie

Spasibo

Premnogo blagodarya